

IDENTITIES FOR THE HURWITZ ZETA FUNCTION, GAMMA FUNCTION, AND L -FUNCTIONS

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ABSTRACT. We derive several identities for the Hurwitz and Riemann zeta functions, the Gamma function, and Dirichlet L -functions. They involve a sequence of polynomials $\alpha_k(s)$ whose study was initiated in [R]. The expansions given here are practical and can be used for the high precision evaluation of these functions, and for deriving formulas for special values. We also present a summation formula and use it to generalize a formula of Hasse.

1. INTRODUCTION

Let $s \in \mathbb{C}$ and define $\alpha_k(s)$ to be the Taylor coefficients given by:

$$\left(\frac{-\log(1-t)}{t} \right)^{s-1} = \sum_0^\infty \alpha_k(s) t^k, \quad |t| < 1. \quad (1.1)$$

Then $\alpha_0(s) = 1$, $\alpha_1(s) = (s-1)/2$, and, in [R], the following formula is proved:

$$\alpha_{k+1}(s) = \frac{1}{k(k+1)(k+2)} \sum_{j=1}^k \frac{\alpha_j(s) j(k+k^2+s(2k+2-j))}{(k-j+1)(k-j+2)}, \quad k \geq 1. \quad (1.2)$$

The above recursion shows that, for $k \geq 1$, $\alpha_k(s)/(s-1)$ is a polynomial in s with positive rational coefficients, and allows one to obtain the bound [R]:

$$|\alpha_k(s)| \leq c_s \frac{(1 + \log(k+1))^{|s|+1}}{k+1}, \quad (1.3)$$

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where

$$c_s = \frac{|s-1|}{|s|+1} (|s|+2)2^{|s|+1}. \quad (1.4)$$

This bound was used in [R] in the derivation of the following formulas for the Riemann zeta and Gamma functions, valid for all $s \in \mathbb{C}$ not a pole of the relevant function:

$$\Gamma(s) = \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s+k}. \quad (1.5)$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s+k-1}. \quad (1.6)$$

For positive integer λ :

$$\zeta(s-\lambda) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=1}^{\lambda} (-1)^{\lambda+j} \frac{j! S(\lambda, j)}{s+k-j-1}. \quad (1.7)$$

where $S(\lambda, j)$ are Stirling numbers of the second kind. Finally,

$$\zeta(s+1) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \Psi_1(s+k), \quad (1.8)$$

with $\Psi_1(s+k)$ the trigamma function.

At the time of writing [R], the author also developed analogous formulas for Dirichlet L -functions via the Hurwitz zeta function, as well as a variety of additional formulas for the Riemann zeta function and the Gamma function, and we report on these here. We collect our main formulas in the theorem below. In equations (1.9)- (1.16), each stated formula is valid for any $s \in \mathbb{C}$ that is not a pole of the lhs.

Theorem 1.1. *Let $\Re w > 0$. Then,*

$$\Gamma(s) = w^s \Gamma(w) \sum_{k=0}^{\infty} \frac{\alpha_k(s) \Gamma(s+k)}{\Gamma(s+k+w)}. \quad (1.9)$$

Specializing to $w = N+1$, a positive integer:

$$\Gamma(s) = (N+1)^s N! \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s+k)(s+k+1) \dots (s+k+N)}. \quad (1.10)$$

For $a > 0$:

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \frac{\Gamma(s+k-1) \Gamma(a)}{\Gamma(s+k+a-1)}. \quad (1.11)$$

For non-negative integer N :

$$\zeta(s) = \sum_1^N n^{-s} + \frac{N!}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s+k-1) \dots (s+k-1+N)}. \quad (1.12)$$

For $a > 0$ and λ a non-negative integer:

$$\zeta(s-\lambda, a) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=0}^{\lambda} \frac{\Gamma(s+k-j-1)\Gamma(a)}{\Gamma(s+k+a-j-1)} c_a(\lambda, j), \quad (1.13)$$

where the coefficients $c_a(\lambda, j)$ are defined by (5.4).

We also have, for positive integer Λ :

$$\sum_{\lambda=1}^{\Lambda} b_{\lambda} \zeta(s-\lambda) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s+k-2) \dots (s+k-\Lambda-1)}, \quad (1.14)$$

where the b_{λ} are given by (6.12).

Let χ be a non-trivial Dirichlet character for the modulus q . Then,

$$L(s, \chi) = \frac{1}{q^s \Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \Gamma(s+k-1) \sum_{m=1}^{q-1} \frac{\chi(m) \Gamma(m/q)}{\Gamma(s+k+m/q-1)}. \quad (1.15)$$

Furthermore, for λ a non-negative integer:

$$\begin{aligned} L(s-\lambda, \chi) &= \frac{1}{\Gamma(s) q^{s-\lambda}} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{m=1}^{q-1} \chi(m) \Gamma(m/q) \\ &\quad \times \sum_{j=0}^{\lambda} \frac{\Gamma(s+k-j-1)}{\Gamma(s+k+m/q-j-1)} c_{m/q}(\lambda, j), \end{aligned} \quad (1.16)$$

where $c_{m/q}(\lambda, j)$ are defined by (5.4)

As a consequence of these formulas, we have several identities for special values. Two formulas for $L(s, \chi)$ at non-positive integers s are summarized in the following theorem.

Theorem 1.2. For non-negative integer r and a non-trivial Dirichlet character χ for the modulus q :

$$L(-r, \chi) = r! q^r \sum_{k=0}^r \frac{(-1)^{k-1} \alpha_k(-r)}{(r+1-k)!} \sum_{m=1}^{q-1} \chi(m) (m/q-1) \dots (m/q+k-r-1). \quad (1.17)$$

and, for non-negative integer λ :

$$\begin{aligned} L(1 - \lambda, \chi) &= q^{\lambda-1} \sum_{j=0}^{\lambda} \left(\frac{(-1)^j}{j!} + \sum_{k=1}^j \frac{(-1)^{k-j}}{(j-k)!} \alpha'_k(1) \right) \\ &\quad \times \sum_{m=1}^{q-1} \chi(m) c_{m/q}(\lambda, j) (m/q - 1) \dots (m/q - j + k). \end{aligned} \quad (1.18)$$

To compute $\alpha'_k(1)$ one can use either of the formulas from [R]:

$$\alpha_{k+1}(1)' = \frac{1}{k+2} - \frac{1}{k+1} \sum_{j=1}^k \frac{j}{k-j+2} \alpha_j(1)', \quad k \geq 0. \quad (1.19)$$

or

$$\alpha'_k(1) = \frac{1}{k} \int_0^1 (x)_k dx. \quad (1.20)$$

We also derive several other formulas for special values. For example,

$$\gamma = \sum_{m=1}^N \frac{1}{m} - \log(N+1) - N! \sum_{k=1}^{\infty} \frac{\alpha_k(0)}{k(k+1) \dots (k+N)}, \quad (1.21)$$

is shown, in Section 4, to follow from (1.10). Equation (1.21) is known and first due to Kluyver [Kl]. In [R], the author attributed his inspiration to Kenter's short note on γ [Ke], but the author has since discovered that Kluyver [Kl] essentially had the same formulas as Kenter for γ , and also (1.21). The above can also be viewed as a more precise form of the formula, often taken as the definition of γ ,

$$\gamma = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{1}{m} - \log(N+1). \quad (1.22)$$

Finally, we derive several interesting formulas that generalize a formula of Hasse.

The alternating zeta function, or Dirichlet eta function, is defined by the Dirichlet series:

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \zeta(s)(1 - 2^{1-s}). \quad (1.23)$$

Theorem 1.3. *For all $s \in \mathbb{C}$, and $\Re s_0 > -1$:*

$$\frac{1}{\Gamma(s_0 + 1)} \sum_{m=0}^{\infty} (m+1) \left(\int_0^{\infty} \frac{x^{s_0} \exp(-x(m+1))}{(1 + \exp(-x))^{m+2}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (1.24)$$

Furthermore, for all $s, s_0 \in \mathbb{C}$:

$$\begin{aligned} \eta(s + s_0) = \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=1}^{m+1} s(m+1, l) \eta(s_0 + 1 - l) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \end{aligned} \quad (1.25)$$

where $s(m+1, l)$ are Stirling numbers of the first kind.

Notice that (1.25) expresses $\eta(s)$, and hence $\zeta(s)$, in terms of its values at any collection of points $s_0, s_0 - 1, s_0 - 2, \dots$.

The special case $s_0 = 0$ in the above theorem simplifies. Substituting $t = \exp(-x)$ into the integral of the first formula,

$$\begin{aligned} (m+1) \int_0^{\infty} \frac{\exp(-x(m+1))}{(1 + \exp(-x))^{m+2}} dx &= (m+1) \int_0^1 \frac{t^m}{(1+t)^{m+2}} dt \\ &= \frac{t^{m+1}}{(1+t)^{m+1}} \Big|_0^1 = \frac{1}{2^{m+1}}, \end{aligned} \quad (1.26)$$

we get

$$\eta(s) = \zeta(s)(1 - 2^{1-s}) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (1.27)$$

a formula first conjectured by Knopp and proved by Hasse[H], later rediscovered by Sondow[S].

A similar formula holds for Dirichlet L -functions for a limited number of Dirichlet characters. For any non-trivial $\chi \bmod q \leq 5$ we have:

$$L(s + s_0, \chi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) L(s_0 + 1 - l, \chi) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (1.28)$$

valid for all $s, s_0 \in \mathbb{C}$. The reason that we need to restrict here to $q \leq 5$ has to do with the location of the q -th roots of unity relative to the point 1, as described in Section 8.4.

1.1. A summation formula. We give several other formulas related to the above. For example, we develop a formula that holds for all non-trivial Dirichlet characters for any modulus. To this end we use the following summation formula which we detail in Section 9.

Let g, h be complex valued functions on the non-negative integers, say increasing at most exponentially, i.e. assume there exists $c \in \mathbb{R}$

such that $g(n), h(n) = O(\exp(cn))$. Define the m -th finite differences recursively:

$$\begin{aligned}\Delta^0 h(j) &= h(j) \\ \Delta^m h(j) &= \Delta^{m-1} h(j+1) - \Delta^{m-1} h(j), \quad m \geq 1.\end{aligned}\quad (1.29)$$

Then

$$\sum_{j=0}^m (-1)^j h(j+1) \binom{m}{j} = (-1)^m \Delta^m h(1). \quad (1.30)$$

Let

$$G(z) := \sum_{n=1}^{\infty} g(n) z^{n-1}. \quad (1.31)$$

We show that

$$\sum_{n=1}^{\infty} g(n) h(n) z^{n-1} = \sum_{m=0}^{\infty} G^{(m)}(z) \Delta^m h(1) \frac{z^m}{m!}, \quad (1.32)$$

holds in some disc centred on $z = 0$.

In some applications, for example to the Hurwitz zeta function, one might prefer to write the n -th term as $g(n)h(n)z^n$ and start the sum at $n = 0$. Thus, for functions g, h on the non-negative integers, growing at most exponentially,

$$\sum_{n=0}^{\infty} g(n) h(n) z^n = \sum_{m=0}^{\infty} \tilde{G}^{(m)}(z) \Delta^m h(0) \frac{z^m}{m!}, \quad (1.33)$$

holds on some disc centred on $z = 0$, where

$$\tilde{G}(z) := \sum_{n=0}^{\infty} g(n) z^n. \quad (1.34)$$

The analytic continuation of the rhs of (1.32) or (1.33) to a given point z depends on the location of the singularities of $G(w)$ or $\tilde{G}(w)$ in relation to the point z , and also on the rate of growth of $\Delta^m h(1)$ or $\Delta^m h(0)$. In our applications, we substitute $z = 1$ after considering the analytic continuation of the function G or \tilde{G} .

As a simple illustrative example, take $g(n) = 1$ if $n = N$ and 0 otherwise, so that $\tilde{G}(z) = g(N)z^N$, and $\tilde{G}^{(m)}(z) = 0$ for all $m > N$. Substituting $z = 1$ we get, as a special case, Newton's forward difference formula:

$$h(N) = h(0) + N\Delta h(0) + \frac{N(N-1)}{2!} \Delta^2 h(0) + \frac{N(N-1)(N-2)}{3!} \Delta^3 h(0) + \dots \quad (1.35)$$

As an application, we give the following identity.

Theorem 1.4. *Let $\lambda > 0$. If $\Re s > 0$, then*

$$\eta(s) = \exp(-\lambda) \sum_{m=0}^{\infty} \lambda^m \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!}, \quad (1.36)$$

the sum over m being uniformly convergent on compact subsets of $\Re s > 0$. Furthermore, as $m \rightarrow \infty$,

$$\sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!} \sim \frac{(-1)^m \exp(-\lambda)}{\lambda^m (m+1+\lambda)^s}, \quad (1.37)$$

uniformly on compact subsets of $\Re s > 0$.

Our summation formula is then applied to the Hurwitz zeta function in Section 9.1 and to Dirichlet L -functions in Section 9.2.

For the Hurwitz zeta function, we actually consider the alternating Hurwitz zeta defined by

$$\zeta^*(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}, \quad (1.38)$$

where $a > 0$, and, in the above series, $\Re s > 0$. Our summation formula gives the following identities, as well as analytic continuation in the stated regions below.

Theorem 1.5. *For $\Re s_0 > -1$ and $s \in \mathbb{C}$,*

$$\begin{aligned} \zeta^*(s + s_0, a) = & \frac{1}{\Gamma(s_0 + 1)} \sum_{m=0}^{\infty} \left(\int_0^{\infty} \frac{x^{s_0} \exp(-x(m+a))(m+a+(a-1)\exp(-x))}{(1+\exp(-x))^{m+1}} dx \right) \\ & \times \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}. \end{aligned} \quad (1.39)$$

For all $s_0, s \in \mathbb{C}$,

$$\zeta^*(s + s_0, a) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \zeta^*(s_0 + 1 - l, a) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}. \quad (1.40)$$

Let $N+1$ be a positive integer. Then, for all $s_0, s \in \mathbb{C}$,

$$\begin{aligned} \zeta^*(s + s_0, N+1) = & \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \zeta^*(s_0 + 1 - l, N+1) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \end{aligned} \quad (1.41)$$

Notice that the second formula above is *not* a specialization of the first. The difference between the above two formulas is in the factor $j + 1$ vs $j + a$ in the sum over j .

For Dirichlet L -functions we have the following theorem, valid for any non-trivial Dirichlet character for any modulus q .

Theorem 1.6. *For $\Re s_0 > -1$, and $s \in \mathbb{C}$,*

$$L(s+s_0, \chi) = \sum_{m=0}^{\infty} (m+1) \left(\int_0^{\infty} \frac{x^{s_0} \exp(-x(m+1))}{(1 + \exp(-x))^{m+2}} dx \right) \sum_{j=0}^m \frac{\chi(j+1) \binom{m}{j}}{(j+1)^s}. \quad (1.42)$$

and, for all $s_0, s \in \mathbb{C}$,

$$L(s+s_0, \chi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \eta(s_0+1-l) \sum_{j=0}^m \frac{\chi(j+1) \binom{m}{j}}{(j+1)^s}. \quad (1.43)$$

Notice that these expansions express $L(s, \chi)$ in terms of the eta function $\eta(s) = \zeta(s)(1 - 2^{1-s})!$

As a special case, set $s_0 = 0$ to get

$$L(s, \chi) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^m \frac{\chi(j+1) \binom{m}{j}}{(j+1)^s}, \quad (1.44)$$

valid for any non-trivial Dirichlet character χ , and all $s \in \mathbb{C}$.

Formulas (1.24)- (1.28) and (1.39)- (1.44) converge uniformly for s_0, s in compact subsets of the regions in which the formulas are claimed to be true. As part of their derivation we give bounds for the finite differences given in each of the sums over j .

Let $\tau(\chi)$ be the Gauss sum

$$\tau(\chi) = \sum_{j=0}^{q-1} \chi(j) e(j/q), \quad (1.45)$$

where

$$e(t) := \exp(2\pi i t). \quad (1.46)$$

We have the following estimates.

Theorem 1.7. *Let $a > 0$. Then, as $m \rightarrow \infty$,*

$$\sum_{j=0}^m \frac{(-1)^j}{(j+a)^s} \binom{m}{j} \sim \frac{\log(m)^{s-1} \Gamma(a)}{m^a \Gamma(s)}, \quad (1.47)$$

uniformly for s in compact subsets of \mathbb{C} . In the event that $s \in \mathbb{Z}$ and $s \leq 0$, then we interpret the \sim to mean equality when $m > |s|$.

Let χ be a non-trivial Dirichlet character for the modulus q . Then as $m/q^2 \rightarrow \infty$

$$\begin{aligned} \sum_{j=0}^m \chi(j+1) \binom{m}{j} &\sim \\ \frac{\tau(\chi)}{q} (e(-1/q)(1 + e(-1/q))^m + \chi(-1)e(1/q)(1 + e(1/q))^m) \\ &\leq \frac{2}{q^{1/2}} C_q^m, \end{aligned} \quad (1.48)$$

and

$$C_q = |1 + e(\pm 1/q)| < 2. \quad (1.49)$$

Furthermore,

$$\begin{aligned} \sum_{j=0}^m \chi(j+1) \binom{m}{j} &\sim \\ = \frac{\tau(\chi)}{q} 2^{m+1} e^{-\frac{m\pi^2}{2q^2}} \times \begin{cases} \cos(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = 1, \\ -i \sin(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = -1. \end{cases} \end{aligned} \quad (1.50)$$

as $m/q^2 \rightarrow \infty$ and $m/q^4 \rightarrow 0$. In (1.48) and (1.50), we interpret \sim to mean equality whenever the rhs vanishes.

We also have, for q fixed and as $m \rightarrow \infty$,

$$\sum_{j=0}^m \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j} \ll m^M C_q^m, \quad (1.51)$$

uniformly for s on compact subsets of $\Re s > -M$, where M is a non-negative integer, and with the implied constant in the \ll also depending on q and M .

These estimates say that the m -th finite differences of $1/(j+a)^s$ and of $(-1)^j \chi(j+1)/(j+1)^s$ are exponentially smaller than the trivial bound obtained by using $\sum_0^m \binom{m}{j} = 2^m$. In the case of Dirichlet characters, the extra power of -1 is crucially needed to get to get such a small bound.

We remark that Coffey, building on the author's paper [R], has separately arrived at formulas (1.9) and (1.11) of Theorem 1.1. His application, however, is to the polygamma function and Stieltjes constants, whereas our focus is on formulas for L -functions and analytic aspects. Furthermore, his derivation is formal in that it does not contain any discussion on the convergence, or rate of convergence, of the sums involved in these formulas, and merely asserts analytic continuation.

Coffey uses, in his paper, the expansion

$$\left(\frac{\ln(x+1)}{x}\right)^z = z \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{B_k^{(z+k)}}{(z+k)}, \quad |x| < 1, \quad (1.52)$$

where the coefficients $B_k^{(z+k)}$ are generalized Bernoulli numbers given by the generating function

$$\left(\frac{z}{e^z - 1}\right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{z^n}{n!}, \quad |z| < 2\pi, \quad (1.53)$$

and remarks that

$$\alpha_k(s) = (s-1) \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)}. \quad (1.54)$$

We regard the generating function (1.1) and the coefficients $\alpha_k(s)$, to be more fundamental than the generating functions (1.53) and (1.52) to the problem at hand.

In this paper we focus our attention on deriving formulas for L -functions of degree 1. It seems, however, that some of our techniques can be applied to higher degree L -functions. We plan to explore higher degree L -functions, and to consider computational aspects of these formulas and variants, in a subsequent paper.

2. EXPANSION FOR THE HURWITZ ZETA FUNCTION

The Hurwitz zeta function is defined, for $\Re s > 1$ and $a > 0$, by

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}. \quad (2.1)$$

We have

$$\begin{aligned} \Gamma(s)\zeta(s, a) &= \int_0^\infty x^{s-1} \sum_{n=0}^{\infty} \exp(-(n+a)x) dx \\ &= \int_0^\infty \frac{x^{s-1} \exp(-ax)}{1 - \exp(-x)} dx. \end{aligned} \quad (2.2)$$

Substituting $t = 1 - \exp(-x)$, the above becomes, on applying (1.1),

$$\begin{aligned} \Gamma(s)\zeta(s, a) &= \int_0^1 (-\log(1-t))^{s-1} (1-t)^{a-1} \frac{dt}{t} \\ &= \sum_{k=0}^{\infty} \alpha_k(s) \int_0^1 t^{s+k-2} (1-t)^{a-1} dt. \end{aligned} \quad (2.3)$$

The rearranging of integration and summation over k is justified by bound (1.3) which shows that, for given s , the sum over k converges uniformly for t in any closed subset of $(0, 1)$. Now

$$\int_0^1 t^{s+k-2}(1-t)^{a-1} = \beta(s+k-1, a) = \frac{\Gamma(s+k-1)\Gamma(a)}{\Gamma(s+k+a-1)}. \quad (2.4)$$

Therefore,

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \frac{\Gamma(s+k-1)\Gamma(a)}{\Gamma(s+k+a-1)}. \quad (2.5)$$

Note that (2.5) combined with the bound (1.3) for $|\alpha_k(s)|$ also provides the meromorphic continuation in s of $\zeta(s, a)$ to all of \mathbb{C} , with the only pole being simple at $s = 1$, because the sum converges uniformly for s in compact sets away from its poles. For the latter fact, we also need

$$\frac{\Gamma(s+k-1)}{\Gamma(s+k+a-1)} \sim \frac{1}{k^a}, \quad (2.6)$$

as $k \rightarrow \infty$ which follows from Gauss' formula (3.6), also expressible as

$$\Gamma(s+N+1) \sim N^s N!. \quad (2.7)$$

Apply the above with N replaced by k , and s replaced, in the numerator of (2.6), by $s-2$, and, in the denominator, by $s+a-2$.

The poles of the numerator of the k -th summand in (2.5) occur at $s = -k+1, -k, -k-1, -k-2, \dots$. All of these, except for the pole at $s = 1$ when $k = 0$, are cancelled by the zeros of $1/\Gamma(s)$ at $s = 0, -1, -2, \dots$, thus the only pole occurs at $s = 1$.

If we let, above, $a = 1$ and use $\Gamma(k+s-1)/\Gamma(s+k) = 1/(s+k-1)$ we recover (1.6). And, if we take $a = N+1$ to be a positive integer, we get

$$\zeta(s) - \sum_1^N n^{-s} = \frac{N!}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s+k-1) \dots (s+k-1+N)}, \quad (2.8)$$

i.e.

$$\zeta(s) = \sum_1^N n^{-s} + \frac{N!}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s+k-1) \dots (s+k-1+N)}. \quad (2.9)$$

We can increase the rate of convergence of this sum by selecting N to be larger. For example, in Table 1 we compare the precision achieved with $s = 3$, $N = 1$, $N = 5$, and $N = 20$, $N = 100$, and $k \leq K$, for

various values of K . More specifically, we truncate the sum at $k \leq K$ and denote the remainder by :

$$R(K, N, s) := \frac{N!}{\Gamma(s)} \sum_{k=K+1}^{\infty} \frac{\alpha_k(s)}{(s+k-1) \dots (s+k-1+N)}. \quad (2.10)$$

In our table, we depict the relative remainder terms

$$\tilde{R}(K, N, s) := R(K, N, s)/\zeta(s), \quad (2.11)$$

for $s = 3$.

K	$\tilde{R}(K, 1)$	$\tilde{R}(K, 5)$	$\tilde{R}(K, 20)$	$\tilde{R}(K, 100)$
0	0.09876701304	0.003735350802	8.885612538e-05	8.054816065e-07
1	0.06410420585	0.001259436003	1.056632936e-05	2.148135338e-08
2	0.04503966189	0.0005029064811	1.595623563e-06	7.505774529e-10
4	0.02594760777	0.0001135959722	6.161196116e-08	1.697998088e-12
8	0.01207251038	1.310916493e-05	3.577149365e-10	4.694895745e-17
16	0.004670251882	7.985644458e-07	2.203486644e-13	1.938746948e-24
32	0.001580885488	2.957659659e-08	1.468535646e-17	1.343384486e-35
64	0.0004908542321	7.867242541e-10	1.559347771e-22	3.318995397e-51
128	0.0001445094596	1.715803026e-11	4.578210941e-28	2.669903582e-71
256	4.117178337e-05	3.342699708e-13	6.093017067e-34	3.364327536e-95
512	1.148739118e-05	6.112998825e-15	5.183860931e-40	7.923516397e-122
1024	3.160289401e-06	1.078325101e-16	3.464372739e-46	3.670183638e-150
2048	8.606974887e-07	1.861800819e-18	2.037483910e-52	1.820228675e-179

TABLE 1. A comparison of the remainder terms $\tilde{R}(K, N, 3)$ for $N = 1, 5, 20, 100$ and various values of K .

Out of curiosity, we compare with the traditional way of computing $\zeta(s)$ using Euler-MacLaurin summation:

$$\begin{aligned} \zeta(s) &= \sum_1^N n^{-s} + \frac{N^{1-s}}{s-1} + \sum_1^K \binom{s+k-2}{k-1} \frac{B_k}{k} N^{-s-k+1} \\ &\quad - \binom{s+K-1}{K} \int_N^\infty B_K(\{t\}) t^{-s-K} dt. \end{aligned} \quad (2.12)$$

where $B_k, B_k(x)$ denotes the Bernoulli numbers and polynomials [R2]. and $B_k := B_k(0)$. Table 2 depicts, for $s = 3$ and the same values of N and K as in Table 1, the relative remainder term $\tilde{R}_2(K, N, s) :=$

$R_2(K, N, s)/\zeta(s)$, where

$$R_2(K, N, s) := -\binom{s+K-1}{K} \int_N^\infty B_K(\{t\}) t^{-s-K} dt, \quad (2.13)$$

K	$\tilde{R}_2(K, 1)$	$\tilde{R}_2(K, 5)$	$\tilde{R}_2(K, 20)$	$\tilde{R}_2(K, 100)$
0	-0.2478610589	-0.002999137452	-5.069543603e-05	4.138739872e-07
1	0.1680926274	0.0003284920385	1.298774753e-06	-2.079699113e-09
2	-0.03988421573	-4.270910495e-06	-1.080516795e-09	6.931868307e-14
4	0.02944139866	1.659288259e-07	2.695929470e-12	-6.931313924e-18
8	0.08490189016	1.233350305e-09	8.380153455e-17	-3.464912761e-25
16	43.97791945	3.248852315e-12	4.036386548e-24	-4.339822180e-38
32	2.000506814e+11	1.894001979e-13	8.399102960e-35	-6.233469067e-60
64	5.607820702e+39	3.383888667e-07	1.661297238e-47	-6.266500089e-95
128	2.810377072e+115	3.618530655e+24	1.065733432e-54	-1.163548513e-146
256	8.493688329e+304	3.878203997e+124	1.480620677e-31	-1.227926982e-212
512	3.186217020e+760	1.702666495e+401	5.605030930e+91	-1.276702408e-268
1024	2.130486030e+1825	1.530602415e+1108	2.919002699e+490	-1.548683921e-227
2048	6.062592955e+4262	7.835225059e+2829	4.672401590e+1595	-5.541412864e+162

TABLE 2. A comparison of the remainder terms $\tilde{R}_2(K, N, 3)$ for $N = 1, 5, 20, 100$ and various values of K .

Formulas (2.9) and (2.12) are different in a number of ways. First, our sum over k converges and truncating the series after K terms gives a better approximation as $K \rightarrow \infty$, while the sum in the Euler-MacLaurin formula provides a divergent asymptotic expansion for $\zeta(s)$. Second, our formulas give the meromorphic continuation of $\zeta(s)$ and of $\zeta(s, a)$ to all of \mathbb{C} with a simple pole with residue 1 at $s = 1$. On the other hand, the Euler-MacLaurin formula only gives, for given K , the meromorphic continuation of $\zeta(s)$ up to $\Re s > -K + 1$.

One can show (see 2.2.5 of [R2]), that the remainder term in the Euler-MacLaurin formula satisfies, for $\sigma := \Re s > -K + 1$,

$$|R_2(K, N, s)| \leq \frac{2\zeta(K)}{N^{\sigma-1}} \frac{|s+K-1|}{\sigma+K-1} \frac{|\Gamma(s+K-1)|}{|\Gamma(s)|(2\pi N)^K}. \quad (2.14)$$

On the other hand, writing the k -th term in (2.10) as

$$\frac{\alpha_k(s)\Gamma(s+k-1)\Gamma(N+1)}{\Gamma(s)\Gamma(s+k+N)} \quad (2.15)$$

we have, for given s and K , that

$$|R(K, N, s)| = O(N^{-\sigma-K}), \quad (2.16)$$

with the implied constant in the big-O depending on s and K . Here we have used the asymptotic formula

$$\frac{\Gamma(N+1)}{\Gamma(s+k+N)} \sim \frac{1}{N^{s+k-1}} \quad (2.17)$$

as $N \rightarrow \infty$, which can be proven using Gauss' formula (3.6) with s replaced by $s+k-1$ or using Stirling's formula. Therefore, in the N -aspect, formula (2.8) is roughly comparable to the Euler-MacLaurin formula, though with different dependence on K and s . The main difference between the two is in the extra $(2\pi)^{-K}$ in (2.14), and the $\alpha_k(s)$, which grows exponentially in $|s|$, in each term of $R(K, N, s)$.

Next we examine the behaviour of $R(K, N, s)$ in the K -aspect. The k -th term of (2.10) is

$$\frac{\alpha_k(s)\Gamma(s+k-1)\Gamma(N+1)}{\Gamma(s)\Gamma(s+k+N)}. \quad (2.18)$$

By (1.3), this equals, for fixed s and N ,

$$O(k^{-N-2+\epsilon}), \quad (2.19)$$

for any $\epsilon > 0$, because $\alpha_k(s) = O(k^{-1+\epsilon})$ and $\Gamma(s+k-1)/\Gamma(s+k+N) = O(k^{-N-1})$. Summing over $k \geq K+1$ thus gives

$$R(K, N, s) = O(K^{-N-1+\epsilon}). \quad (2.20)$$

with the implied constant depending on s and N .

3. A RELATED EXPANSION FOR THE GAMMA FUNCTION

Let $\Re w, \Re s > 0$ and consider

$$\begin{aligned} \Gamma(s)w^{-s} &= \int_0^\infty x^{s-1}e^{-wx}dx \\ &= \int_0^1 (-\log(1-t)/t)^{s-1}t^{s-1}(1-t)^{w-1}dt \\ &= \sum_{k=0}^\infty \alpha_k(s) \int_0^1 t^{s+k-1}(1-t)^{w-1}dt \\ &= \sum_{k=0}^\infty \alpha_k(s) \frac{\Gamma(s+k)\Gamma(w)}{\Gamma(s+k+w)}. \end{aligned} \quad (3.1)$$

Therefore,

$$\Gamma(s) = w^s \Gamma(w) \sum_{k=0}^\infty \frac{\alpha_k(s)\Gamma(s+k)}{\Gamma(s+k+w)}. \quad (3.2)$$

By (1.3), this series converges uniformly for all s in a given compact set away from its poles and therefore provides the meromorphic continuation of $\Gamma(s)$. Specializing to $w = N + 1$, where N is a non-negative integer, we get

$$\Gamma(s) = (N + 1)^s N! \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s + k)(s + k + 1) \dots (s + k + N)}. \quad (3.3)$$

Notice that, as $N \rightarrow \infty$, the $k = 0$ term dominates. We can see this by writing the k -th term of the sum as:

$$\frac{\alpha_k(s)}{(s + k)} \frac{1}{s(s + 1) \dots (s + N)} \frac{s(s + 1) \dots (s + k - 1)}{(s + N + 1) \dots (s + N + k)}. \quad (3.4)$$

Now, $s/(s + N + 1) = O(1/N)$, with the implied constant depending on s . Furthermore, if N is sufficiently large in comparison to $|\Re s|$, then $|(s + j)/(s + N + j + 1)| < 1$ for all positive integers j . Thus the terms with $k \geq 1$ contribute

$$\ll \frac{(N + 1)^s N!}{s(s + 1) \dots (s + N)} \frac{1}{N} \sum_{k=1}^{\infty} \frac{|\alpha_k(s)|}{|s + k|} \quad (3.5)$$

to (3.3). Bound (1.3) shows that the sum over k converges, hence the above is $O(1/N)$ times the $k = 0$ term. We thus get Gauss' formula for $\Gamma(s)$:

$$\Gamma(s) = \lim_{N \rightarrow \infty} \frac{N^s N!}{s(s + 1) \dots (s + N)}. \quad (3.6)$$

Here we have also used $(N + 1)^s \sim N^s$ as $N \rightarrow \infty$. Thus, the $k = 0$ term in formula (3.3) connects Euler's formula for the Gamma function, i.e. our starting point, to that of Gauss.

4. FORMULA FOR EULER'S CONSTANT

In equation (3.3), subtract $1/s$ from both sides and then let $s \rightarrow 0$. On the lhs, we get $-\gamma$, i.e. the negative of Euler's constant (here, we can define $-\gamma$ to be the constant term in the Laurent series about $s = 0$ of $\Gamma(s)$). On the rhs, the $1/s$ will cancel with a portion of the $k = 0$ term. More precisely, expand $(N + 1)^s$ as:

$$(N + 1)^s = 1 + \log(N + 1)s + O(s^2). \quad (4.1)$$

Furthermore, the $k = 0$ term, with the $N!$ but not the $(N + 1)^s$, equals

$$\frac{N!}{s(s + 1) \dots (s + N)} = \sum_{m=0}^N \frac{(-1)^m}{s + m} \binom{N}{m}, \quad (4.2)$$

the latter from the partial fraction expansion (6.1). Therefore, multiplying (4.1), with (3.3), using the above for $k = 0$, collecting terms, and taking the limit as $s \rightarrow 0$, we have

$$\gamma = \sum_{m=1}^N \frac{(-1)^{m+1}}{m} \binom{N}{m} - \log(N+1) - N! \sum_{k=1}^{\infty} \frac{\alpha_k(0)}{k(k+1) \dots (k+N)}. \quad (4.3)$$

Now, by the binomial theorem,

$$\sum_{m=1}^N \frac{(-1)^{m+1}}{m} \binom{N}{m} = - \int_{-1}^0 \frac{((1+t)^N - 1)}{t} dt. \quad (4.4)$$

Substituting $u = 1+t$, expanding $(u^N - 1)/(u - 1) = 1 + u + \dots u^{N-1}$, and integrating termwise gives

$$\sum_{m=1}^N \frac{(-1)^{m+1}}{m} \binom{N}{m} = 1 + 1/2 + \dots 1/N. \quad (4.5)$$

Hence

$$\gamma = \sum_{m=1}^N \frac{1}{m} - \log(N+1) - N! \sum_{k=1}^{\infty} \frac{\alpha_k(0)}{k(k+1) \dots (k+N)}. \quad (4.6)$$

5. HURWITZ ZETA FUNCTION SHIFTED

Next we describe a formula for $\zeta(s - \lambda, \beta)$ where λ is positive integer and $a > 0$. Begin with

$$\begin{aligned} \Gamma(s) \zeta(s - \lambda, a) &= \int_0^{\infty} x^{s-1} \sum_{n=0}^{\infty} (n+a)^{\lambda} \exp(-(n+a)x) dx \\ &= \int_0^1 (-\log(1-t))^{s-1} \sum_{n=0}^{\infty} (n+a)^{\lambda} (1-t)^{n+a-1} dt. \end{aligned} \quad (5.1)$$

We can express

$$\sum_{n=0}^{\infty} (n+a)^{\lambda} (1-t)^{n+a-1} \quad (5.2)$$

as a rational function in t by starting with the $\lambda = 0$ case, $\sum_{n=0}^{\infty} (1-t)^{n+a-1} = (1-t)^{a-1}/t$, and repeatedly multiplying by $1-t$ and applying $-d/dt$. We can thus prove, inductively, that

$$\sum_{n=0}^{\infty} (n+a)^{\lambda} (1-t)^{n+a-1} = (1-t)^{a-1} \sum_{j=0}^{\lambda} \frac{c_a(\lambda, j)}{t^{j+1}}, \quad (5.3)$$

where $c_a(0, 0) = 1$, $c_a(0, 1) = 0$, and

$$c_a(\lambda + 1, j) = (a - j - 1)c_a(\lambda, j) + jc_a(\lambda, j - 1), \quad (5.4)$$

for $\lambda \geq 1$, $0 \leq j \leq \lambda + 1$ (for the recursion, we also set $c_a(\lambda, \lambda + 1) = 0$). We need not worry about defining $c_a(\lambda, j - 1)$ when $j = 0$ because of the factor of j that appears in front of the $c_a(\lambda, j - 1)$. As usual, we write $(-\log(1 - t))^{s-1} = \sum_{k=0}^{\infty} \alpha_k(s) t^{k+s-1}$, and get

$$\zeta(s - \lambda, a) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{j=0}^{\lambda} \frac{\Gamma(s + k - j - 1) \Gamma(a)}{\Gamma(s + k + a - j - 1)} c_a(\lambda, j). \quad (5.5)$$

When $a = 1$ we have

$$c_1(\lambda, j) = (-1)^{\lambda+j} j! S(\lambda, j) \quad (5.6)$$

and (5.5) can be seen as a hybrid of (1.7) and (2.5).

6. A LINEAR COMBINATION OF ζ THAT CONVERGES QUICKLY

Let Λ be a positive integer. In this section we derive a formula that expresses a linear combination of $\zeta(s - 1), \dots, \zeta(s - \Lambda)$ in a series of a nature similar to (2.9).

We will exploit the fact that the same coefficients $\alpha_k(s)$ appear in (1.7) independent of λ , and that the denominators in (1.7) are particularly simple. By taking linear combinations of $\Gamma(s)\zeta(s - \lambda)$ we can develop a sum whose terms converge more rapidly. We do so by using the partial fraction expansion:

$$\frac{1}{z(z - 1) \dots (z - m)} = \frac{a_0}{z} + \frac{a_1}{z - 1} + \dots + \frac{a_m}{z - m}, \quad (6.1)$$

where

$$a_l = \frac{(-1)^{m-l}}{l!(m-l)!} = \frac{(-1)^{m-l}}{m!} \binom{m}{l}. \quad (6.2)$$

The formula for a_l can be derived by considering the residue of both sides of (6.1) at $z = l$.

Therefore, letting Λ be a positive integer, we wish to find a linear combination:

$$\sum_{\lambda=1}^{\Lambda} b_{\lambda} \zeta(s - \lambda) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{\lambda=1}^{\Lambda} \sum_{j=1}^{\lambda} (-1)^{\lambda+j} \frac{j! S(\lambda, j)}{s + k - j - 1} b_{\lambda} \quad (6.3)$$

such that

$$\sum_{\lambda=j}^{\Lambda} (-1)^{\lambda+j} S(\lambda, j) j! b_{\lambda} = \frac{(-1)^{\Lambda-j}}{(\Lambda - 1)!} \binom{\Lambda - 1}{j - 1}, \quad (6.4)$$

λ	j	$c_a(\lambda, j)$
0	0	1
1	0	$a - 1$
1	1	1
2	0	$(a - 1)^2$
2	1	$2a - 3$
2	2	2
3	0	$(a - 1)^3$
3	1	$3a^2 - 9a + 7$
3	2	$6a - 12$
3	3	6
4	0	$(a - 1)^4$
4	1	$4a^3 - 18a^2 + 28a - 15$
4	2	$12a^2 - 48a + 50$
4	3	$24a - 60$
4	4	24
5	0	$(a - 1)^5$
5	1	$5a^4 - 30a^3 + 70a^2 - 75a + 31$
5	2	$20a^3 - 120a^2 + 250a - 180$
5	3	$60a^2 - 300a + 390$
5	4	$120a - 360$
5	5	120
6	0	$(a - 1)^6$
6	1	$6a^5 - 45a^4 + 140a^3 - 225a^2 + 186a - 63$
6	2	$30a^4 - 240a^3 + 750a^2 - 1080a + 602$
6	3	$120a^3 - 900a^2 + 2340a - 2100$
6	4	$360a^2 - 2160a + 3360$
6	5	$720a - 2520$
6	6	720
7	0	$(a - 1)^7$
7	1	$7a^6 - 63a^5 + 245a^4 - 525a^3 + 651a^2 - 441a + 127$
7	2	$42a^5 - 420a^4 + 1750a^3 - 3780a^2 + 4214a - 1932$
7	3	$210a^4 - 2100a^3 + 8190a^2 - 14700a + 10206$
7	4	$840a^3 - 7560a^2 + 23520a - 25200$
7	5	$2520a^2 - 17640a + 31920$
7	6	$5040a - 20160$
7	7	5040

TABLE 3. A table of $c_a(\lambda, j)$.

for each $1 \leq j \leq \Lambda$. Doing so would then result, by (6.1), in the formula:

$$\sum_{\lambda=1}^{\Lambda} b_{\lambda} \zeta(s - \lambda) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{(s + k - 2) \dots (s + k - \Lambda - 1)}. \quad (6.5)$$

We now determine b_λ so that (6.3) holds. Notice that (6.4), for $1 \leq j \leq \Lambda$, can be written as the matrix equation

$$DS \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_\Lambda \end{pmatrix} = \frac{(-1)^{\Lambda-1}}{(\Lambda-1)!} \begin{pmatrix} \binom{\Lambda-1}{0} \\ -\binom{\Lambda-1}{1} \\ \vdots \\ (-1)^{\Lambda-1} \binom{\Lambda-1}{\Lambda-1} \end{pmatrix}, \quad (6.6)$$

where S is the upper triangular matrix

$$S = ((-1)^{i+j} S(j, i))_{\Lambda \times \Lambda}, \quad (6.7)$$

and D is the diagonal matrix

$$D = \begin{pmatrix} 1! & & & \\ & 2! & & \\ & & 3! & \\ & & & \ddots \\ & & & & \Lambda! \end{pmatrix}. \quad (6.8)$$

Thus,

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_\Lambda \end{pmatrix} = \frac{(-1)^{\Lambda-1}}{(\Lambda-1)!} S^{-1} D^{-1} \begin{pmatrix} \binom{\Lambda-1}{0} \\ -\binom{\Lambda-1}{1} \\ \vdots \\ (-1)^{\Lambda-1} \binom{\Lambda-1}{\Lambda-1} \end{pmatrix}, \quad (6.9)$$

with

$$D^{-1} = \begin{pmatrix} 1! & & & \\ & 1/2! & & \\ & & 1/3! & \\ & & & \ddots \\ & & & & 1/\Lambda! \end{pmatrix} \quad (6.10)$$

and

$$S^{-1} = ((-1)^{i+j} s(j, i))_{\Lambda \times \Lambda}, \quad (6.11)$$

where $s(j, i)$ are the Stirling numbers of the first kind. Therefore, the solution to equation (6.4) is given by:

$$b_\lambda = \frac{(-1)^{\Lambda+\lambda}}{(\Lambda-1)!} \sum_{j=\lambda}^{\Lambda} \frac{s(j, \lambda)}{j!} \binom{\Lambda-1}{\lambda-1}, \quad (6.12)$$

thus yielding (6.5).

7. EXPANSION FOR DIRICHLET L -FUNCTIONS

Let χ be a non-trivial Dirichlet character for the modulus q . Let $L(s, \chi)$ be the Dirichlet L -function

$$L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s}. \quad (7.1)$$

The defining sum is absolutely convergent in $\Re s > 1$ and, if χ is not the trivial character mod q , conditionally convergent for $\Re s > 0$. Rearranging the sum, for $\Re s > 1$, we can write it as

$$\frac{1}{q^s} \sum_{m=1}^{q-1} \chi(m) \zeta(s, m/q). \quad (7.2)$$

Applying (2.5) and changing order of summation gives

$$L(s, \chi) = \frac{1}{\Gamma(s)} \frac{1}{q^s} \sum_{k=0}^{\infty} \alpha_k(s) \Gamma(s+k-1) \sum_{m=1}^{q-1} \frac{\chi(m) \Gamma(m/q)}{\Gamma(s+k+m/q-1)}. \quad (7.3)$$

Similarly, for any positive integer q ,

$$\Gamma(s) \zeta(s) = \frac{1}{q^s} \sum_{k=0}^{\infty} \alpha_k(s) \Gamma(s+k-1) \sum_{m=1}^q \frac{\Gamma(m/q)}{\Gamma(s+k+m/q-1)}. \quad (7.4)$$

Notice that the sum over m in (7.4) runs up to $m \leq q$ rather than $q-1$ since, without the factor $\chi(m)$, we cannot ignore the $m = q$ term.

Next, if we substitute $s = 1$ into (7.3) then only the $k = 0$ term contributes because $\alpha_k(1) = 0$ if $k \geq 1$, and $\Gamma(s+k-1)/\Gamma(s+k+m/q-1)$ does not have a pole at $s = 1$ when $k \geq 1$.

To deal with the $k = 0$ term, consider, for $a > 0$, the Laurent series about $s = 1$

$$\frac{\Gamma(s-1)}{\Gamma(s+a-1)} = \frac{1}{\Gamma(a)(s-1)} - \frac{\gamma + \psi(a)}{\Gamma(a)} + O(s-1), \quad (7.5)$$

where

$$\psi(a) := \frac{\Gamma'(a)}{\Gamma(a)}. \quad (7.6)$$

Substituting this, with $a = m/q$, into the $k = 0$ term of (7.3), summing over m and using $\sum \chi(m) = 0$, so that the terms involving $1/(s-1)$ and γ sum to zero, and then letting $s = 1$, we get the well known (see, for example, Proposition 10.2.5 of [C]) formula

$$L(1, \chi) = -\frac{1}{q} \sum_{m=1}^{q-1} \chi(m) \frac{\Gamma'(m/q)}{\Gamma(m/q)}. \quad (7.7)$$

Next, let r be a non-negative integer. Consider the residue of (7.3) at $s = -r$ which involves, on the rhs, the terms with $k = 0, \dots, r+1$. Using the fact that the residue at $s = -r$ of $\Gamma(s+k-1)$ is equal to $(-1)^{r-k+1}/(r-k+1)!$, we have that

$$L(-r, \chi) = r!q^r \sum_{k=0}^{r+1} \frac{(-1)^{k-1} \alpha_k(-r)}{(r+1-k)!} \sum_{m=1}^{q-1} \frac{\chi(m) \Gamma(m/q)}{\Gamma(m/q+k-r-1)}. \quad (7.8)$$

We can simplify slightly using $\Gamma(m/q)/\Gamma(m/q-r+k-1) = (m/q-1) \dots (m/q-r+k-1)$ for $k < r+1$ and 1 for $k = r+1$. The latter case does not contribute when summed against $\chi(m)$, and we get

$$L(-r, \chi) = r!q^r \sum_{k=0}^r \frac{(-1)^{k-1} \alpha_k(-r)}{(r+1-k)!} \sum_{m=1}^{q-1} \chi(m) (m/q-1) \dots (m/q+k-r-1). \quad (7.9)$$

For example, when $r = 0$ the above gives

$$L(0, \chi) = - \sum_{m=1}^{q-1} \chi(m) (m/q-1) = -\frac{1}{q} \sum_{m=1}^{q-1} \chi(m) m. \quad (7.10)$$

The coefficients $\alpha_k(-r)$ that are needed for a given r in (7.9) are easily computed using, for instance, the recurrence (1.2).

We can write a similar formula for $\zeta(-r)$, though, without the $\chi(m)$, we cannot ignore the $k = r+1$ term, and, furthermore, the sum over m needs to be up to $m \leq q$:

$$\zeta(-r) = r!q^r \sum_{k=0}^{r+1} \frac{(-1)^{k-1} \alpha_k(-r)}{(r+1-k)!} \sum_{m=1}^q (m/q-1) \dots (m/q+k-r-1). \quad (7.11)$$

The innermost summand above is taken to equal 1 if $k = r+1$. For example, if $q = 1$, we get

$$\zeta(-r) = (-1)^r r! \alpha_{r+1}(-r). \quad (7.12)$$

In, Section 7 of [R], the author showed that $\alpha_{r+1}(-r) = B_{r+1}/(r+1)!$. Thus, the above gives Euler's formula $\zeta(-r) = (-1)^r B_{r+1}/(r+1)$.

Next, we develop a second formula for $L(s, \chi)$ evaluated at negative integers. Using (5.5) we have, for positive integer λ :

$$\begin{aligned} L(s - \lambda, \chi) &= \frac{1}{\Gamma(s)} \frac{1}{q^{s-\lambda}} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{m=1}^{q-1} \chi(m) \Gamma(m/q) \\ &\times \sum_{j=0}^{\lambda} \frac{\Gamma(s + k - j - 1)}{\Gamma(s + k + m/q - j - 1)} c_{m/q}(\lambda, j). \end{aligned} \quad (7.13)$$

Substituting $s = 1$, we get contributions from the terms $k = 0$, and from $j = k$ with $j = 1, \dots, \lambda$. The latter is accounted for by the pole of $\Gamma(s + k - j - 1)$ balancing out against the zero of $\alpha_k(s)$ at $s = 1$. Thus,

$$\begin{aligned} L(1 - \lambda, \chi) &= q^{\lambda-1} \sum_{j=0}^{\lambda} \left(\frac{(-1)^j}{j!} + \sum_{k=1}^j \frac{(-1)^{k-j}}{(j-k)!} \alpha'_k(1) \right) \\ &\times \sum_{m=1}^{q-1} \chi(m) c_{m/q}(\lambda, j) (m/q - 1) \dots (m/q - j + k). \end{aligned} \quad (7.14)$$

8. GENERAL DIRICHLET SERIES

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \quad (8.1)$$

be a Dirichlet series, assumed to be absolutely convergent for $\Re s > \sigma_1$.

Substitute formula (3.1), with $w = n$ into the Dirichlet series, and change order of summation over n and k :

$$\begin{aligned} L(s) &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} b(n) \sum_{k=0}^{\infty} \frac{\alpha_k(s) \Gamma(s+k) (n-1)!}{\Gamma(s+k+n)} \\ &= \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \alpha_k(s) \sum_{n=1}^{\infty} b(n) \frac{\Gamma(s+k) (n-1)!}{\Gamma(s+k+n)}. \end{aligned} \quad (8.2)$$

Changing order of the sums can be justified for $\Re s > \sigma_1$, but we do not do so here, because our emphasis below will mainly be on the first equality, rather than the second. However, before turning our attention to the first equality, we note that, if $b(n)$ is identically equal to 1 for

$n \geq N + 1$ and 0 otherwise, then the inner sum over n in the last line above equals

$$\frac{\Gamma(s + k - 1)N!}{\Gamma(s + k + N)} \quad (8.3)$$

This can be seen by taking the identity

$$\int_0^1 t^{s+k-1} \sum_{n=N+1}^{\infty} (1-t)^{n-1} dt = \sum_{n=N+1}^{\infty} \frac{\Gamma(s+k)\Gamma(n)}{\Gamma(s+k+n)}, \quad (8.4)$$

and summing the geometric series on the lhs. Thus, in this case, equation (8.2) reduces to (2.9). The fact that we can simplify the sum over n is what makes (8.2) particularly useful in the case of the Riemann and Hurwitz zeta functions, and Dirichlet L -functions.

Next we explore the first equality of (8.2). Using the partial fraction expansion (6.1), we have

$$\frac{(n-1)!}{\Gamma(s+k+n)} = \sum_{m=0}^{n-1} \frac{(-1)^m}{s+k+m} \binom{n-1}{m}. \quad (8.5)$$

Therefore,

$$L(s) = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} b(n) \sum_{k=0}^{\infty} \alpha_k(s) \sum_{m=0}^{n-1} \frac{(-1)^m}{s+k+m} \binom{n-1}{m}. \quad (8.6)$$

But bound (1.3) allows us to rearrange the two inner sums over k and m :

$$\sum_{k=0}^{\infty} \alpha_k(s) \sum_{m=0}^{n-1} \frac{(-1)^m}{s+k+m} \binom{n-1}{m} = \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s+k+m}. \quad (8.7)$$

The resulting inner sum over k equals

$$\begin{aligned} \int_0^1 (-\log(1-t)/t)^{s-1} t^{s+m-1} dt &= \int_0^{\infty} x^{s-1} \exp(-x) (1 - \exp(-x))^m dx \\ &= \Gamma(s) \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s} \binom{m}{j}, \end{aligned} \quad (8.8)$$

which we see using (1.1), then changing variable $t = 1 - \exp(-x)$, expanding $(1 - \exp(-x))^m$ using the binomial theorem, and integrating termwise. We thus have the identity

$$\frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\alpha_k(s)}{s+k+m} = \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s} \binom{m}{j}. \quad (8.9)$$

Notice that, while in the integrals in (8.8) we require $\Re s > -m$, both sides of the above equation are entire functions of s , and hence, by analytic continuation, equality holds for all s .

Hence,

$$L(s) = \sum_{n=1}^{\infty} b(n) \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.10)$$

Since the inner double sum is just another way to express $1/n^s$, formula (8.10) holds for all s for which the defining Dirichlet series converges conditionally.

We can also arrive at the above more directly by starting with the binomial coefficient identity

$$\sum_{m=j}^{n-1} (-1)^m \binom{n-1}{m} \binom{m}{j} = \begin{cases} (-1)^{n-1}, & \text{if } j = n-1, \\ 0, & \text{otherwise,} \end{cases} \quad (8.11)$$

for $j \geq 0$, and $n \geq 1$. To prove this identity, consider the coefficient of z^{n-1} in $(z-1)^{n-1} \sum_{m=j}^{\infty} \binom{m}{j} z^m = (-1)^{n-1} z^j (1-z)^{n-j-2}$.

Thus

$$\sum_{j=0}^{n-1} \frac{(-1)^j}{(j+1)^s} \sum_{m=j}^{n-1} (-1)^m \binom{n-1}{m} \binom{m}{j} = \frac{1}{n^s}, \quad (8.12)$$

because only the $j = n-1$ terms survives with an extra $(-1)^n$ introduced from the inner sum. Substituting into the Dirichlet series for $L(s)$, and rearranging the inner double sum gives (8.10).

8.1. Heuristic manipulations. Next we proceed heuristically, but we will justify our resulting formulas in certain cases in the next section.

Changing order of summation, and ignoring for the moment the fact that the resulting sums over n diverge unless $b(n)$ is rapidly decreasing, gives

$$L(s) = \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \sum_{n=1}^{\infty} b(n) (n-1) \dots (n-m) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.13)$$

We interpret the typically divergent sum over n as being equal to the m -th derivative, evaluated at $z = 1$, of the analytic continuation, if it exists, of the series

$$f(z) := \sum_{n=1}^{\infty} b(n) z^{n-1}, \quad (8.14)$$

thus suggesting

$$L(s) = \sum_{m=0}^{n-1} \frac{(-1)^m f^{(m)}(1)}{m!} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.15)$$

For example, let $L(s) = \zeta(s)(1 - 2/2^s) = 1 - 2^{-s} + 3^{-3} - 4^{-s} + \dots$ be the alternating zeta function. We consider here the alternating zeta function, rather than ζ itself, because the pole of $\zeta(s) = 1$ creates a slight complication that we wish to circumvent. The factor $(1 - 2/2^s)$ has a zero at $s = 1$, balancing the pole of $\zeta(s)$. Thus, $\zeta(s)(1 - 2/2^s)$ extends to an entire function. Then $f(z) = 1 - z + z^2 - z^3 + \dots = 1/(1+z)$, and $f^{(m)}(1) = (-1)^m m!/2^{m+1}$, hence (8.13) reduces to

$$\zeta(s)(1 - 2/2^s) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (8.16)$$

i.e. to Hasse's formula described in the introduction.

We can also, in (8.13), expand

$$(n-1) \dots (n-m) = \frac{1}{n} n(n-1) \dots (n-m) = \frac{1}{n} \sum_{l=0}^{m+1} s(m+1, l) n^l. \quad (8.17)$$

Here, we have multiplied and divided, in the first equality, by n so as to have, in the second equality, a polynomial in n rather than in $n-1$. Substitute the rhs of (8.17) into (8.13), and rearrange the sum over n and l to get

$$L(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-l}} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.18)$$

We interpret the typically divergent sum $\sum_{n=1}^{\infty} b(n) n^{l-1}$ as being equal to $L(1-l)$, suggesting the following interpolation formula for $L(s)$:

$$L(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) L(1-l) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.19)$$

Now, $s(m+1, 0) = 0$ for all $m \geq 0$, and thus we can start the sum at $l = 1$ unless $L(s)$ has a pole at $s = 1$.

Consider again the case $L(s) = \zeta(s)(1 - 2/2^s)$. In this case, for $l \geq 1$,

$$L(1-l) = \zeta(1-l)(1 - 2^l) = (-1)^{l-1} (1 - 2^l) B_l / l, \quad (8.20)$$

and the sum over m in (8.19) equals

$$\sum_{l=1}^{m+1} (-1)^{l-1} (1 - 2^l) s(m+1, l) B_l / l. \quad (8.21)$$

Comparing with (8.16) suggests that the above equals

$$(-1)^m m! / 2^{m+1}. \quad (8.22)$$

In the next section we will rigorously prove the above manipulations in a few cases, including the alternating zeta function, and thus yielding equality between (8.21) and (8.22).

Note that we can also consider the above manipulations for the Dirichlet series with coefficients $b(n)/n^{s_0}$ for some $s_0 \in \mathbb{C}$. Writing

$$L(s + s_0) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s_0}} \frac{1}{n^s} \quad (8.23)$$

and letting

$$L(s_0, z) := \sum_{n=1}^{\infty} \frac{b(n)}{n^{s_0}} z^{n-1}, \quad (8.24)$$

we anticipate (in certain cases) two formulas:

$$L(s + s_0) = \sum_{m=0}^{n-1} \frac{(-1)^m L^{(m)}(s_0, 1)}{m!} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (8.25)$$

where $L^{(m)}(s_0, 1)$ is the m -th derivative with respect to z of $L(s_0, z)$ evaluated at $z = 1$, and

$$L(s + s_0) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) L(s_0 + 1 - l) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.26)$$

8.2. Justifying the manipulations. We are able to prove the formulas of the previous section, specifically equations (8.25) and (8.26) for the alternating zeta and alternating Hurwitz zeta functions, and Dirichlet L -functions $L(s, \chi)$ where χ is non-trivial character for the modulus $q \leq 5$. In Section 9.2 we develop a related summation formula that can be used to give interesting formulas valid for all q .

For higher degree L -functions, for example, degree 2 L -functions associated to cusp forms of given weight, level, and character, it appears from numerical experiments that formula (8.26) holds but with some adjustment- additional convergence producing terms, arising from a smoothed approximate functional equation, seem to be needed. We will revisit the issue of higher degree L -functions in a future paper.

We start by modifying equation (8.10) by introducing an extra parameter z . Let

$$L(s, z) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s} z^{n-1}. \quad (8.27)$$

Assume that there exists a real number r such that

$$b(n) \ll n^r, \quad (8.28)$$

so that the coefficients $b(n)$ do not grow too quickly, and the series defining $L(s, z)$ converges for all $s \in \mathbb{C}$ and $|z| < 1$. Then, as before, replacing $1/n^s$ by (8.12), we rigorously have

$$L(s, z) = \sum_{n=1}^{\infty} b(n) z^{n-1} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.29)$$

Now, for $|z| < 1/2$, the sum over n and m converges absolutely. That is because (1.47) shows that the sum over j is bounded, in the m -aspect, by an amount which is generously $O(1)$, with the implied constant uniform for s in compact subsets of \mathbb{C} . Therefore, the absolute value of the rhs of the above is, on using $\sum_{m=0}^{n-1} \binom{n-1}{m} = 2^{n-1}$,

$$\ll_s \sum_{n=1}^{\infty} |b(n)| |2z|^{n-1}. \quad (8.30)$$

This converges for all $|z| < 1/2$ because $b(n)$ is assumed to grow at most polynomially in n . Therefore, for $|z| < 1/2$, and $s \in \mathbb{C}$, we have on rearranging the sums over m and n and pulling out a $z^m/m!$ from the sum over n :

$$L(s, z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!} \sum_{n=m+1}^{\infty} b(n) (n-1) \dots (n-m) z^{n-m-1} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.31)$$

Our next goal is to obtain the analytic continuation in z of the sum over n and to substitute $z = 1$. In general, this cannot be done, but in some examples we are able to do so.

8.3. Alternating zeta function. The alternating zeta function is defined to be

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \quad (8.32)$$

We wish to allow shifts, so we let $b(n) = (-1)^{n-1}/n^{s_0}$ for some $s_0 \in \mathbb{C}$. Thus we are considering

$$\eta(s + s_0, z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{s_0}} \frac{z^{n-1}}{n^s}. \quad (8.33)$$

This can also be regarded as a special case of the Lerch zeta function. We have introduced the parameter s_0 since we wish to examine the general expansions of the form considered in (8.26).

In this case, the sum over n in (8.31) reduces to

$$\sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{n^{s_0}} (n-1) \dots (n-m) z^{n-m-1}. \quad (8.34)$$

Now

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{s_0}} z^{n-1} = \frac{1}{\Gamma(s_0)} \int_0^{\infty} \frac{x^{s_0-1} \exp(-x)}{1+z \exp(-x)} dx, \quad (8.35)$$

which we can prove by expanding the denominator as a geometric series and integrating termwise. The above integral converges and is analytic in both variables for $\Re s_0 > 0$ and $z \notin (-\infty, -1]$. Differentiating the lhs termwise m times within its disc of convergence gives (8.34). Hence differentiating the rhs m times yields

$$\begin{aligned} & \sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{n^{s_0}} (n-1) \dots (n-m) z^{n-m-1} \\ &= \frac{(-1)^m m!}{\Gamma(s_0)} \int_0^{\infty} \frac{x^{s_0-1} \exp(-x(m+1))}{(1+z \exp(-x))^{m+1}} dx. \end{aligned} \quad (8.36)$$

Also note that we can bound these derivatives using knowledge of the radius of convergence of the Taylor series, about the point $w = z$ of the function $\eta(s_0, w)$. Consider the Taylor expansion

$$\eta(s_0, w) = \sum_{m=0}^{\infty} \frac{\eta^{(m)}(s_0, z)}{m!} (w-z)^m. \quad (8.37)$$

Here $\eta^{(m)}$ refers to the m -th derivative of η with respect to the second variable. Assume that $\Re z > -1$, so that the nearest singularity of the rhs of (8.35) is the point -1 . Thus, the above converges absolutely for all $|w-z| < |1+z|$, in particular, at $w = 0$ if $|z| < |1+z|$. The latter holds when $\Re z > -1/2$ (as can be seen by drawing z and $1+z$).

Thus, returning to (8.31) (with $L(s) = \eta(s)$), for $\Re z > -1/2$, $\Re s_0 > 0$, $s \in \mathbb{C}$,

$$\eta(s+s_0, z) = \frac{1}{\Gamma(s_0)} \sum_{m=0}^{\infty} z^m \left(\int_0^{\infty} \frac{x^{s_0-1} \exp(-x(m+1))}{(1+z \exp(-x))^{m+1}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \quad (8.38)$$

(recall, for the purpose of convergence, that the sum over j is $O_s(1)$). Therefore, substituting $z = 1$ we get

$$\eta(s+s_0) = \frac{1}{\Gamma(s_0)} \sum_{m=0}^{\infty} \left(\int_0^{\infty} \frac{x^{s_0-1} \exp(-x(m+1))}{(1+\exp(-x))^{m+1}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (8.39)$$

thus justifying (8.25) in the case of $L(s) = \eta(s)$. For any given s_0 satisfying $\Re s_0 > 0$, the rhs above converges uniformly in s on compact subsets of \mathbb{C} , thus also giving the analytic continuation of the alternating zeta function.

We can extend the validity of (8.39) in the s_0 aspect by integrating by parts

$$\int_0^\infty \frac{x^{s_0-1} \exp(-x(m+1))}{(1+\exp(-x))^{m+1}} dx = \frac{(m+1)}{s_0} \int_0^\infty \frac{x^{s_0} \exp(-x(m+1))}{(1+\exp(-x))^{m+2}} dx. \quad (8.40)$$

While the integral on the lhs is convergent for $\Re s_0 > 0$, the integral on the right converges for $\Re s_0 > -1$. Therefore, for $\Re s_0 > -1$ and $s \in \mathbb{C}$,

$$\begin{aligned} \eta(s + s_0) &= \\ \frac{1}{\Gamma(s_0 + 1)} \sum_{m=0}^\infty (m+1) \left(\int_0^\infty \frac{x^{s_0} \exp(-x(m+1))}{(1+\exp(-x))^{m+2}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \end{aligned} \quad (8.41)$$

For example, substituting $s_0 = 0$ gives Hasse's formula (1.27) as described in the introduction.

Because the m -th term decreases exponentially in m for given s_0 (with $\Re s_0 > -1$), and uniformly for s in compact subsets of \mathbb{C} , equation (8.41) also gives the analytic continuation of η .

The above also shows that, for *any* $s \in \mathbb{C}$ and $\Re s_0 > -1$, the limit, as $z \rightarrow 1$ of the rhs of (8.33) exists and is equal to $\eta(s + s_0)$ as expressed, for example, in (8.41). Therefore, returning to (8.34) and expanding using (8.17) we get

$$\begin{aligned} \eta(s + s_0) &= \\ \sum_{m=0}^\infty \frac{(-1)^m}{m!} \sum_{l=1}^{m+1} s(m+1, l) \eta(s_0 + 1 - l) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \end{aligned} \quad (8.42)$$

While we initially assumed $\Re s_0 > -1$, we can extend (8.41) to $\Re s_0 > -M$, where M is any positive integer, by repeatedly integrating by parts. Therefore, equation (8.42) holds for all $s_0, s \in \mathbb{C}$.

8.4. Dirichlet L -functions. Let χ be a non-trivial Dirichlet character for the modulus q . Define

$$L(s_0, z, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^{s_0}} z^{n-1}, \quad |z| < 1. \quad (8.43)$$

Now, for $\Re s_0 > 0$,

$$L(s_0, z, \chi) = \frac{1}{\Gamma(s_0)} \int_0^\infty \frac{x^{s_0-1} \sum_{a=1}^{q-1} \chi(a) z^{a-1} \exp(-ax)}{1 - z^q \exp(-qx)} dx, \quad (8.44)$$

which we can see by expanding the denominator as a geometric series and using the periodicity mod q of χ . Note that, for given x , the singularity of the integrand at $z = \exp(x)$ is removable because the numerator vanishes at $z = \exp(x)$. Therefore, the rhs is analytic in some neighbourhood, depending on q , of $z = 1$.

As in the discussion concerning η , we can get an estimate for the size of the m -th derivative with respect to z of $L(s_0, z, \chi)$, evaluated at $z = 1$ by considering the singularities of the rhs of (8.44) which occur along rays emanating outward from the non-one q -th roots of unity, i.e. at $z = r \exp(2\pi i j/q)$, $r > 1$, $1 \leq j < q$.

We require that the distance from the point 1 to the nearest singularity, $\exp(\pm 2\pi i/q)$ be greater than 1 so that the sum analogous to (8.38) continues analytically to $z = 1$. This occurs when $q < 5$ (at $q = 6$, $\exp(2\pi i/6)$ has a distance of 1 from the point 1, and for $q > 6$ the closest root is even closer). Thus, for any non-trivial $\chi \bmod q \leq 5$ we have:

$$L(s + s_0, \chi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) L(s_0 + 1 - l) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}, \quad (8.45)$$

valid for all $s, s_0 \in \mathbb{C}$.

9. A SUMMATION FORMULA

Next we consider a generalization of (8.12). The same binomial coefficient identity which gave that formula for $1/n^s$ can be used for any function $h(n)$ on the positive integers:

$$h(n) = \sum_{j=0}^{n-1} (-1)^j h(j+1) \sum_{m=j}^{n-1} (-1)^m \binom{n-1}{m} \binom{m}{j}, \quad (9.1)$$

because only the $j = n-1$ term survives. Therefore rearranging sums,

$$h(n) = \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \sum_{j=0}^m (-1)^j h(j+1) \binom{m}{j}. \quad (9.2)$$

Let $g, h : \mathbb{N} \rightarrow \mathbb{C}$ be two functions on the positive integers. Apply the above identity for $h(n)$ to the power series

$$\sum_{n=1}^{\infty} g(n)h(n)z^{n-1}, \quad (9.3)$$

and then rearrange the sum over m and n , pulling out a $1/m!$ and z^m from the sum over m :

$$\begin{aligned} & \sum_{n=1}^{\infty} g(n)h(n)z^{n-1} \\ &= \sum_{m=0}^{\infty} \frac{(-z)^m}{m!} \sum_{n=m+1}^{\infty} g(n)(n-1)\dots(n-m)z^{n-m-1} \\ & \quad \times \sum_{j=0}^m (-1)^j h(j+1) \binom{m}{j}. \end{aligned} \quad (9.4)$$

The latter step of rearranging the sums over m and n can be justified for sufficiently small z assuming modest bounds on g, h . For example, assuming that $g(n)$ and $h(n)$ are $O(n^r)$ for some real number r , and using trivial bounds on the sums of binomial coefficients, the sum over m and n converges absolutely for $|z| < 1/4$. By trivial, we mean using $\sum_{j=0}^m \binom{m}{j} = 2^m$, and $\sum_{m=0}^{n-1} \binom{n-1}{m} = 2^{n-1}$. One can even assume that g, h grow exponentially, i.e. $O(\exp(cn))$ for some $c \in \mathbb{R}$, and still get an identity valid on some disc centred on $z = 0$.

We can view the sum over n as being the m -th derivative with respect to z of the power series

$$G(z) := \sum_{n=1}^{\infty} g(n)z^{n-1}. \quad (9.5)$$

Thus equation (9.4) can be written as:

$$\sum_{n=1}^{\infty} g(n)h(n)z^{n-1} = \sum_{m=0}^{\infty} G^{(m)}(z) \frac{(-z)^m}{m!} \sum_{j=0}^m (-1)^j h(j+1) \binom{m}{j}. \quad (9.6)$$

Note that, in the previous sections, with the choice $h(n) = 1/n^s$, and the asymptotic formula (1.47), so that $\sum_{j=0}^m (-1)^j \binom{m}{j} / (j+1)^s \ll_s 1$, we were thus allowed to initially take $|z| < 1/2$. The second ingredient for η and $L(s, \chi)$, with $g(n) = (-1)^{n-1}/n^{s_0}$ or $\chi(n)/n^{s_0}$, was to obtain the analytic continuation of the rhs of (9.4) to a neighbourhood of the point $z = 1$. We achieved this by expressing the appropriate $G(z)$ in

closed form in (8.35) and (8.44) from which we were able to deduce the rate of decay of $G^{(m)}(z)$.

The inner sum over j ,

$$\sum_{j=0}^m (-1)^j h(j+1) \binom{m}{j}, \quad (9.7)$$

can be interpreted in terms of the finite difference operator. Define

$$\begin{aligned} \Delta h(j) &= h(j+1) - h(j) \\ \Delta^m h(j) &= \Delta^{m-1} h(j+1) - \Delta^{m-1} h(j). \end{aligned} \quad (9.8)$$

Then (9.7) is equal to

$$(-1)^m \Delta^m h(1), \quad (9.9)$$

hence

$$\sum_{n=1}^{\infty} g(n) h(n) z^{n-1} = \sum_{m=0}^{\infty} G^{(m)}(z) \Delta^m h(1) \frac{z^m}{m!}. \quad (9.10)$$

Note that, because our application is to Dirichlet series, we prefer, above, to write the n -th term of the lhs as $g(n)h(n)z^{n-1}$. However, in other applications, as to the Hurwitz zeta function below, one might prefer to write the n -th term as $g(n)h(n)z^n$ and start the sum at $n = 0$. Thus, as a variant, for functions g, h on the non-negative integers

$$\sum_{n=0}^{\infty} g(n) h(n) z^n = \sum_{m=0}^{\infty} \tilde{G}^{(m)}(z) \Delta^m h(0) \frac{z^m}{m!}, \quad (9.11)$$

where

$$\tilde{G}(z) := \sum_{n=0}^{\infty} g(n) z^n. \quad (9.12)$$

The range of validity in z of (9.10) and (9.11) depends on the rates of decay of $g(n)$, and $\Delta^m h(0)$ as discussed above, for example being valid on some disc centred at 0 if $g(n)$ and $h(n)$ grow at most exponentially in n . The analytic continuation of the rhs to a given point z depends further on the location of the singularities of $G(w)$ or of $\tilde{G}(w)$ in relation to the point z .

To illustrate our summation formula, let $\lambda > 0$, and take $g(n) = (-1)^{n-1} \lambda^n$, $h(n) = n^{-s} \lambda^{-n}$. Then $G(z) = \lambda/(1+\lambda z)$, and $G^{(m)}(1)(-1)^m/m! = \lambda^{m+1}/(1+\lambda^{m+1})$. Formally, this choice of g and h gives Amore's generalization of Hasse's identity [A]:

$$\eta(s) = \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{(1+\lambda)^{m+1}} \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s \lambda^{j+1}}. \quad (9.13)$$

In order to justify the above, one also needs to obtain a rate of decay for the sum over j (depending on λ), and this is done in [A].

Many variants are possible. For example, let $\lambda > 0$, $g(n) = (-1)^{n-1} \lambda^n / (n-1)!$, $h(n) = n^{-s} \lambda^{-n} (n-1)!$. Then $G(z) = \exp(-\lambda z)$, and $G^{(m)}(1) = (-\lambda)^m \exp(-\lambda)$. Therefore, substituting into our summation formula, and simplifying the factorials and binomial coefficient that appear, we get

$$\eta(s) = \exp(-\lambda) \sum_{m=0}^{\infty} \lambda^m \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!}. \quad (9.14)$$

We justify this formula for $\Re s > 0$. Note that, while $h(n)$ grows very quickly with n , to justify the above formula what we need, for the purposes of convergence and rearranging sums as in (9.4), is an estimate for the size of the sum

$$\sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!}. \quad (9.15)$$

For $\Re s > 0$, this sum is equal to

$$\frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \sum_{j=0}^m \frac{(-1)^j \exp(-(j+1)x)}{\lambda^j (m-j)!}. \quad (9.16)$$

Note that, as $m \rightarrow \infty$, the main contribution to the above integral arises from smaller x , which we can see by breaking the integral into, say, $\int_0^1 + \int_1^{\infty}$, and on the second integral, breaking the sum into $0 < j \leq m/2$ and $m/2 < j \leq m$. We thus truncate the integral at $x = 1$, and then reverse the sum over j to get, as $m \rightarrow \infty$,

$$\begin{aligned} \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!} &\sim \\ \frac{(-1)^m}{\lambda^m \Gamma(s)} \int_0^1 x^{s-1} \exp(-(m+1)x) \sum_{j=0}^m \frac{(-\lambda)^j \exp(jx)}{j!}. \end{aligned} \quad (9.17)$$

The resulting sum in the integrand is the truncation of the Maclaurin series for $\exp(y)$, with $y = -\lambda \exp(x)$, i.e.

$$\sum_{j=0}^m \frac{(-\lambda)^j \exp(jx)}{j!} = \exp(-\lambda \exp(x)) - \sum_{j=m+1}^{\infty} \frac{(-\lambda)^j \exp(jx)}{j!}. \quad (9.18)$$

An analysis shows that, for given $\lambda > 0$ and as $m \rightarrow \infty$, that the latter sum does not contribute to the main asymptotics of the integral. Therefore, dropping that sum, and then extending the domain of

integration to ∞ , we have

$$\begin{aligned}
& \sum_{j=0}^m \frac{(-1)^j}{(j+1)^s \lambda^j (m-j)!} \\
& \sim \frac{(-1)^m}{\Gamma(s) \lambda^m} \int_0^\infty x^{s-1} \exp(-\lambda \exp(x) - (m+1)x) dx \\
& \sim \frac{(-1)^m \exp(-\lambda)}{\lambda^m (m+1+\lambda)^s}, \tag{9.19}
\end{aligned}$$

with all our estimates above uniform on compact subsets of $\Re s > 0$, thus justifying equation (9.14) in that half-plane. The last \sim can be derived by substituting $t = (m+1)x$ into the integral and expanding, for $t = O(m+1)$,

$$\lambda \exp(t/(m+1)) = \lambda + \lambda t/(m+1) + O(\lambda t^2/m^2). \tag{9.20}$$

A more careful analysis shows for a given λ , that to get a good approximation (i.e. within a specified small relative error ϵ), for a given λ , we need $m \gg \lambda$, with the implied constant depending on s and ϵ .

Formula (1.36) becomes interesting as we take λ large, because the terms in that formula are, for m sufficiently large and including the $\exp(-\lambda)$ that appears in front of the sum,

$$\sim \frac{(-1)^m \exp(-2\lambda)}{(m+1+\lambda)^s}. \tag{9.21}$$

Thus, while $(-1)^m/(m+1+\lambda)^s$ decays at roughly the same rate as terms in the Dirichlet series defining $\eta(s)$, our estimate shows that the terms in (9.14) are eventually, for $m \gg_s \lambda$, exponentially small. Thus, for λ large, there is a transition zone, as we sum over m , in which we quickly achieve roughly $2\lambda/\log(10)$ digits accuracy for $\eta(s)$ before the sum continues its leisurely convergence to $\eta(s)$.

9.1. Application to the alternating Hurwitz zeta function. We define, for $a, \Re s > 0$, the alternating Hurwitz zeta function to be

$$\zeta^*(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}. \tag{9.22}$$

Setting, in (9.11), $g(n) = (-1)^n/(n+a)^{s_0}$ and $h(n) = 1/(n+a)^s$, we have

$$\begin{aligned}\tilde{G}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^{s_0}} z^n \\ &= \frac{1}{\Gamma(s_0)} \int_0^{\infty} x^{s_0-1} \sum_{n=0}^{\infty} (-z)^n \exp(-(n+a)x) dx \\ &= \frac{1}{\Gamma(s_0)} \int_0^{\infty} \frac{x^{s_0-1} \exp(-ax)}{1+z \exp(-x)} dx,\end{aligned}\tag{9.23}$$

and so

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^{s_0}} \frac{1}{(n+a)^s} z^n = \\ &\frac{1}{\Gamma(s_0)} \sum_{m=0}^{\infty} z^m \left(\int_0^{\infty} \frac{x^{s_0-1} \exp(-x(m+a))}{(1+z \exp(-x))^{m+1}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}.\end{aligned}\tag{9.24}$$

We remark that, here, the function $\tilde{G}(z)$ is equal to $\Phi(-z, s_0, a)$, where Φ is the Lerch zeta function.

A similar analysis of (9.23) to that of (8.35) can be carried out and shows that (9.24) holds for $s \in \mathbb{C}$, $\Re s_0 > 0$ and $z \notin (-\infty, -1]$. The asymptotic formula (1.47) provides a uniform bound for the sum over j .

Thus, substituting $z = 1$ gives

$$\begin{aligned}\zeta^*(s + s_0, a) &= \\ &\frac{1}{\Gamma(s_0)} \sum_{m=0}^{\infty} \left(\int_0^{\infty} \frac{x^{s_0-1} \exp(-x(m+a))}{(1 + \exp(-x))^{m+1}} dx \right) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}.\end{aligned}\tag{9.25}$$

We can extend the range of validity to $\Re s_0 > -M$ by repeatedly integrating by parts M times. For example, for $\Re s_0 > -1$ and $s \in \mathbb{C}$:

$$\begin{aligned}\zeta^*(s + s_0, a) &= \\ &\frac{1}{\Gamma(s_0 + 1)} \sum_{m=0}^{\infty} \left(\int_0^{\infty} \frac{x^{s_0} \exp(-x(m+a))(m+a + (a-1)\exp(-x))}{(1 + \exp(-x))^{m+1}} dx \right) \\ &\quad \times \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}.\end{aligned}\tag{9.26}$$

Substituting $s_0 = 0$, the above integral can easily be evaluated to equal $1/2^{m+1}$, thus specializing to another formula of Hasse.

We also have, as in Section 8.3, for all $s_0, s \in \mathbb{C}$,

$$\zeta^*(s + s_0, a) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \zeta^*(s_0 + 1 - l, a) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+a)^s}. \quad (9.27)$$

We give yet another formula for the alternating Hurwitz zeta function, when $a = N + 1$ is positive integer. Taking, in our summation formula (9.4), $g(n) = 0$ if $n \leq N$, and $(-1)^{n-N-1}/n^{s_0}$ if $n \geq N + 1$, and $h(n) = 1/n^s$, we have:

$$\begin{aligned} G(z) &= \sum_1^{\infty} g(n) z^{n-1} = (-1)^N \sum_{N+1}^{\infty} \frac{(-z)^{n-1}}{n^{s_0}} \\ &= \frac{z^N}{\Gamma(s_0)} \int_0^{\infty} x^{s_0-1} \frac{\exp(-(N+1)x)}{1 + z \exp(-x)} dx. \end{aligned} \quad (9.28)$$

This gives the analytic continuation of $G(z)$ to the point $z = 1$ where G has radius of convergence 2. Therefore, substituting $z = 1$ and proceeding as before

$$\begin{aligned} \zeta^*(s + s_0, N + 1) &= \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \zeta^*(s_0 + 1 - l, N + 1) \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{(j+1)^s}. \end{aligned} \quad (9.29)$$

9.2. Dirichlet L -functions again. Next we give formulas that express $L(s, \chi)$, for *any* non-trivial character χ for the modulus q (i.e. without restriction on q), in terms of the Riemann zeta function, or, more precisely, in terms of the alternating zeta function.

We let $g(n) = (-1)^{n-1}/n^{s_0}$ and $h(n) = (-1)^{n-1}\chi(n)/n^s$. Therefore, $G(z)$ here is identical to the function in (8.35) and the whole discussion of (8.32) regarding the function $G(z)$ carries through. We also require a bound for the sum over j , which we will show beats the trivial bound exponentially.

Let

$$C_q := |1 + e(1/q)|. \quad (9.30)$$

Assume that $\Re s > -M$ where M is a non-negative integer. We will prove, for q fixed and as $m \rightarrow \infty$,

$$\sum_{j=0}^m \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j} \ll m^M C_q^m, \quad (9.31)$$

uniformly for s on compact subsets of $\Re s > -M$ (with the implied constant in the \ll also depending on q and M). But $C_q < 2$, hence, dividing by 2^m shows that $\frac{1}{2^m} \sum_{j=0}^m \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j}$ decreases exponentially fast as $m \rightarrow \infty$. It is crucial for this bound, and also for getting a nice rate of decay on $G^{(m)}(z)/m!$, that we include here the factor $(-1)^{n-1}$ in $g(n)$ and $h(n)$.

Thus:

$$L(s+s_0, \chi) = \sum_{m=0}^{\infty} (m+1) \left(\int_0^{\infty} \frac{x^{s_0} \exp(-x(m+1))}{(1 + \exp(-x))^{m+2}} dx \right) \sum_{j=0}^m \frac{\chi(j+1) \binom{m}{j}}{(j+1)^s}. \quad (9.32)$$

and

$$L(s + s_0, \chi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{l=0}^{m+1} s(m+1, l) \eta(s_0 + 1 - l) \sum_{j=0}^m \frac{\chi(j+1) \binom{m}{j}}{(j+1)^s}. \quad (9.33)$$

Equation (9.32) converges for $\Re s_0 > -1$, and $s \in \mathbb{C}$, while (9.33) converges for all $s_0, s \in \mathbb{C}$, uniformly for s_0, s on compact subsets of \mathbb{C} , thus also giving the analytic continuation of $L(s, \chi)$.

9.3. A bound on $\sum_0^m \chi(j+1) \binom{m}{j} / (j+1)^s$. In this section we use the notation

$$e(t) := \exp(2\pi i t). \quad (9.34)$$

We have the following identity

$$\begin{aligned} & \sum_0^m \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j} \\ &= \frac{1}{q} \sum_{l=0}^{q-1} \sum_{j=0}^m \frac{e(-jl/q)}{(j+1)^s} \binom{m}{j} \sum_{a=1}^q \chi(a) e((a-1)l/q), \end{aligned} \quad (9.35)$$

which is easily verified by noting that, for given a , that only the terms j with $j+1 = a \pmod q$ survive the sum over l . Each a , above, thus contributes

$$\sum_{\substack{0 \leq j \leq m \\ j+1 \equiv a \pmod m}} \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j}. \quad (9.36)$$

Summing over all $1 \leq a \leq q$ thus gives the lhs of (9.35).

We first examine the $s = 0$ case because we can obtain very precise information in that case. Identity (9.35) reads

$$\begin{aligned} & \sum_{j=0}^m \chi(j+1) \binom{m}{j} \\ &= \frac{1}{q} \sum_{l=0}^{q-1} (1 + e(-l/q))^m \sum_{a=1}^q \chi(a) e((a-1)l/q), \end{aligned} \quad (9.37)$$

Notice, crucially, the sum over a on the rhs vanishes when $l = 0$. It is for this reason that we are able to beat the trivial bound.

Furthermore, $|1 + e(-l/q)| < 2$ for $1 \leq l \leq q-1$, with the largest values being when $l = 1, q-1$. Also note that, above, the sum over a can be expressed in terms of the Gauss sum. When $l = 1$ the sum over a equals $e(-1/q)\tau(\chi)$, and when $l = q-1$ the sum equals $e(1/q)\chi(-1)\tau(\chi)$.

Therefore, for fixed q ,

$$\begin{aligned} & \sum_{j=0}^m \chi(j+1) \binom{m}{j} \sim \\ & \frac{\tau(\chi)}{q} (e(-1/q)(1 + e(-1/q))^m + \chi(-1)e(1/q)(1 + e(1/q))^m) \\ & \leq \frac{2}{q^{1/2}} C_q^m, \end{aligned} \quad (9.38)$$

where $C_q = |1 + e(\pm 1/q)| < 2$. Note that we have also used $|\tau(\chi)| \leq q^{1/2}$, with equality when χ is primitive. Dividing by 2^m , the rhs decays exponentially in m . Therefore, because $G(z)$ here is given by (8.35) and has Taylor series with radius of convergence 2 at $z = 1$, we can substitute $z = 1$ in our summation formula thus yielding equations (9.32) and (9.33), when $s = 0$.

We can be a bit more precise about the rate of decay of the middle expression of (9.38). Expand

$$1 + e(t) = 2(1 + \pi it - \pi^2 t^2 + \frac{2i}{3} \pi^3 t^3 + O(t^4)). \quad (9.39)$$

The bracketed term can be written as

$$\exp(\pi it - \pi^2 t^2/2 + O(t^4)). \quad (9.40)$$

Therefore,

$$\begin{aligned}
& \frac{\tau(\chi)}{q} (e(-1/q)(1 + e(-1/q))^m + \chi(-1)e(1/q)(1 + e(1/q))^m) \\
&= \frac{\tau(\chi)}{q} 2^m e^{-\frac{m\pi^2}{2q^2}(1+O(1/q^2))} (e^{-\frac{\pi i}{q}(m+2)} + \chi(-1)e^{\frac{\pi i}{q}(m+2)}) \\
&= \frac{\tau(\chi)}{q} 2^{m+1} e^{-\frac{m\pi^2}{2q^2}(1+O(1/q^2))} \times \begin{cases} \cos(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = 1, \\ -i \sin(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = -1. \end{cases}
\end{aligned} \tag{9.41}$$

Note that in the case that χ is a real primitive character, the above simplifies further to

$$\frac{2^{m+1}}{q^{1/2}} e^{-\frac{m\pi^2}{2q^2}(1+O(1/q^2))} \times \begin{cases} \cos(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = 1, \\ \sin(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = -1. \end{cases} \tag{9.42}$$

We can similarly bound the contribution from the terms $2 \leq l \leq q-2$ in (9.37). For simplicity assume that χ is primitive so that the inner sum in (9.37) can be written in terms of $\tau(\chi)$ and is, in absolute value, equal to $q^{1/2}$ or 0, depending whether $\gcd(l, q) = 1$ or not. Furthermore, amongst $2 \leq l \leq q-2$, the maximum $|1 + e(l/q)|^m$ occurs when $l = 2$ or $q-2$. Thus the contribution from $2 \leq l \leq q-2$ is bounded by

$$2^m q^{1/2} e^{-\frac{2m\pi^2}{q^2}(1+O(1/q^2))}, \tag{9.43}$$

i.e. the *rate* of decay in m for the remainder term is roughly 4 times larger.

We have thus shown, for primitive χ , that

$$\begin{aligned}
& \sum_0^m \chi(j+1) \binom{m}{j} \\
&= \frac{\tau(\chi)}{q} 2^{m+1} e^{-\frac{m\pi^2}{2q^2}(1+O(1/q^2))} \times \begin{cases} \cos(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = 1, \\ -i \sin(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = -1. \end{cases} \\
&+ R(\chi, m),
\end{aligned} \tag{9.44}$$

where the remainder $R(\chi, m)$ satisfies

$$R(\chi, m) < 2^m q^{1/2} e^{-\frac{2m\pi^2}{q^2}(1+O(1/q^2))}. \tag{9.45}$$

With respect to uniform asymptotics allowing both q and m to grow, equation (9.44) provides an asymptotic formula for the sum on the left

as $m/q^2 \rightarrow \infty$ and $m/q^4 \rightarrow 0$,

$$\begin{aligned} \sum_{j=0}^m \chi(j+1) \binom{m}{j} &\sim \\ &= \frac{\tau(\chi)}{q} 2^{m+1} e^{-\frac{m\pi^2}{2q^2}} \times \begin{cases} \cos(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = 1, \\ -i \sin(\frac{\pi}{q}(m+2)), & \text{if } \chi(-1) = -1, \end{cases} \end{aligned} \quad (9.46)$$

while the second line of (9.38) provides an asymptotic formula valid in the larger region $m/q^2 \rightarrow \infty$. Note that we do not need to assume that χ is primitive in the above asymptotic formula- that assumption was made to simplify the discussion regarding $R(\chi, m)$, but a similar analysis applies.

We interpret the above asymptotic formula to be an equality in the event that the rhs vanishes. There are two cases to consider. If $\chi(-1) = -1$, then the rhs vanishes iff $m = -2 \pmod{q}$. On the other hand, the terms j and $m-j$ on the lhs cancel each other out, assuming that $j \neq m-j$, because the binomial coefficients match, and, furthermore, when $m = -2 \pmod{q}$, $\chi(m-j+1) = \chi(-2-j+1) = \chi(-1)\chi(j+1) = -\chi(j+1)$. Special care is needed when m is even and $j = m/2$ because then there is just one binomial coefficient, and not a pair. But, if $j = m/2$ then $\chi(j+1) = \chi((m+2)/2)$. However, $m+2$ is assumed, here, to be a multiple of q , hence $\gcd((m+2)/2, q) > 1$ (because $q > 2$ when χ is non-trivial), and so $\chi((m+2)/2) = 0$. Thus, $\chi(j+1) = 0$, and the middle binomial coefficient does not contribute to the sum.

The case $\chi(-1) = 1$ is a bit more complicated, but the idea is the same, namely to show that the terms j and $m-j$ cancel. We have $\cos(\frac{\pi}{q}(m+2)) = 0$ iff $m+2 = rq/2$ where r is an odd integer, i.e. $2m+4 = rq$. Thus q is even because r is odd, and, in fact, divisible by at least 4 because there are no even primitive characters for the modulus 2. Hence $m = -2 \pmod{q/2}$. Because r is odd, we can write it as $r = 2r_0 + 1$. Now, $\chi(m-j+1) = \chi((2r_0+1)q/2 - j - 1) = \chi(q/2 - j - 1) = \chi(j+1 - q/2)$, the last step because $\chi(-1) = 1$.

Next, factor q into distinct prime powers, $q = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$, with $p_1 = 2$, and let $\chi(n) = \chi_{p_1^{\beta_1}}(n) \dots \chi_{p_k^{\beta_k}}(n)$. Now, for $1 < l \leq k$, $\chi_{p_l^{\beta_l}}(j+1 - q/2) = \chi_{p_l^{\beta_l}}(j+1)$, because $q/2 = 0 \pmod{p_l^{\beta_l}}$ for $l > 1$. Thus, to prove here that $\chi(m-j+1) = -\chi(j+1)$, i.e. that the terms j and $m-j$ cancel, we need to show that

$$\chi_{2^{\beta_1}}(j+1 - q/2) = -\chi_{2^{\beta_1}}(j+1) \quad (9.47)$$

for all j and any primitive character mod 2^{β_1} .

Note that we can assume that $j+1$ is odd because $q/2$ is even. Thus, if $j+1$ is even then both $\chi_{2^{\beta_1}}(j+1-q/2)$ and $\chi_{2^{\beta_1}}(j+1)$ are equal to 0.

Write $q = 2^{\beta_1}(2q_0 + 1)$, i.e. write the odd part of q as $2q_0 + 1$. Thus, $\chi_{2^{\beta_1}}(j+1-q/2) = \chi_{2^{\beta_1}}(j+1-2^{\beta_1-1})$.

Now, if $\beta_1 = 2$, then $\chi_4(j+1) = -\chi_4(j-1)$ and we are done. Next, if $\beta_1 \geq 3$ then any character for the modulus $2^{\beta_1} \geq 8$ can be written, for odd n , as

$$\chi_{2^{\beta_1}}(n) = e\left(\frac{\mu\nu}{2} + \frac{\mu'\nu'}{2^{\beta_1-2}}\right) \quad (9.48)$$

where $0 \leq \mu < 2$, $0 \leq \mu' < 2^{\beta_1-2}$, and

$$n \equiv (-1)^\nu 5^{\nu'} \pmod{2^{\beta_1}}, \quad (9.49)$$

with $0 \leq \nu < 2$, $0 \leq \nu' < 2^{\beta_1-2}$. See Chapters 4-5 of [D]. Furthermore, $\chi_{2^{\beta_1}}$ is primitive implies that μ' is odd.

We claim that

$$n - 2^{\beta_1-1} \equiv n(2^{\beta_1-1} + 1) \pmod{2^{\beta_1}} \quad (9.50)$$

which holds because n is assumed to be odd, and

$$5^{2^{\beta_1-3}} \equiv 2^{\beta_1-1} + 1 \pmod{2^{\beta_1}}, \quad (9.51)$$

which can be proven inductively by repeatedly squaring.

Therefore

$$\chi_{2^{\beta_1}}(n - 2^{\beta_1-1}) = \chi(n)\chi(2^{\beta_1-1} + 1) = \chi(n)\chi(5^{2^{\beta_1-3}}). \quad (9.52)$$

But

$$\chi_{2^{\beta_1}}(5^{2^{\beta_1-3}}) = e\left(\frac{\mu'2^{\beta_1-3}}{2^{\beta_1-2}}\right) = \exp(\pi i \mu') = -1, \quad (9.53)$$

the last step because μ' is odd when $\chi_{2^{\beta_1}}$ is primitive.

We have therefore shown that if: χ is primitive, $\chi(-1) = 1$, and $\cos(\frac{\pi}{q}(m+2)) = 0$, then $\chi(j+1) = -\chi(m-j+1)$ for all j . Finally, we need to address, as before, the case where m is even and $j = m/2$, i.e. when the middle binomial coefficient is not part of pair. But, in that case, $\chi(j+1) = \chi((m+2)/2) = \chi(rq/2)$ which equals 0 because q is divisible by 4 and hence $q/2$ is even.

9.3.1. *Bounding the sum in general.* Assume for now that $\Re s > 0$.

We consider the sum over j in (9.35), which we can write as

$$\begin{aligned} & \sum_{j=0}^m \frac{e(-jl/q)}{(j+1)^s} \binom{m}{j} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (1 + \exp(-x)e(-l/q))^m \exp(-x) dx. \end{aligned} \quad (9.54)$$

Similar to the $s = 0$ case, one can show that the $l = 1, q - 1$ terms dominate (the $l = 0$ term in (9.35) vanishes), and we have, for fixed q and as $m \rightarrow \infty$,

$$\begin{aligned} & \sum_{j=0}^m \frac{\chi(j+1)}{(j+1)^s} \binom{m}{j} \sim \\ & \frac{\tau(\chi)}{q\Gamma(s)} \left(e(-1/q) \int_0^\infty x^{s-1} (1 + \exp(-x)e(-1/q))^m \exp(-x) dx \right. \\ & \left. + \chi(-1)e(1/q) \int_0^\infty x^{s-1} (1 + \exp(-x)e(1/q))^m \exp(-x) dx \right). \end{aligned} \quad (9.55)$$

Taking absolute value and using $|1 + \exp(-x)e(-1/q)| \leq C_q$ gives (1.51)

Repeated integration by parts of (9.54), say M times, allows us to obtain the same sort of bound for $\Re s > -M$, at a cost of, on integrating by parts, introducing extra powers of m into the integrand, thus giving bound (1.51).

This rate of decay, beating 2^m by an exponentially small amount, justifies formulas (9.32) and (9.33).

10. AN ESTIMATE FOR $\sum_{j=0}^m \frac{(-1)^j}{(j+a)^s} \binom{m}{j}$

For $a > 0$, we derive an asymptotic formula for

$$\sum_{j=0}^m \frac{(-1)^j}{(j+a)^s} \binom{m}{j}, \quad (10.1)$$

as $m \rightarrow \infty$. The above equals, for $m + \Re s > 0$,

$$\begin{aligned} & \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \exp(-ax) (1 - \exp(-x))^m dx = \\ & \frac{1}{m^a \Gamma(s)} \int_0^m (-\log(v/m))^{s-1} v^{a-1} (1 - v/m)^m dv. \end{aligned} \quad (10.2)$$

Here we have substituted $v = m \exp(-x)$. Now, we can write

$$\log(m/v)^{s-1} = \log m^{s-1} ((1 - \log v / \log m)^{s-1} - 1 + 1), \quad (10.3)$$

so that (10.2) equals

$$\begin{aligned} & \frac{\log(m)^{s-1}}{m^a \Gamma(s)} \int_0^m v^{a-1} (1 - v/m)^m dv + \\ & \frac{\log(m)^{s-1}}{m^a \Gamma(s)} \int_0^m ((1 - \log v / \log m)^{s-1} - 1) v^{a-1} (1 - v/m)^m dv. \end{aligned} \quad (10.4)$$

It is a simple exercise to show that the first integral above is asymptotically equal to $\Gamma(a)$ as $m \rightarrow \infty$ by breaking up the integral into, say, $\int_0^{m^{1/4}} + \int_{m^{1/4}}^m$, and using:

$$\begin{aligned} (1 - v/m)^m &= \exp(m \log(1 - v/m)) \\ &= \exp\left(-v - \frac{v^2}{2m} - \frac{v^3}{3m} - \dots\right), \quad |v| < m \\ &= \exp(-v)(1 + O(1/m^{1/2})), \quad |v| < m^{1/4}. \end{aligned} \quad (10.5)$$

We can also show that the second integral in (10.4) does not contribute to the main term. First, undo our change of variable and express the integral as

$$\int_0^\infty ((x/\log m)^{s-1} - 1)(m \exp(-x))^a (1 - \exp(-x))^m dx.$$

We need to show that this tends to 0 as $m \rightarrow \infty$, and uniformly for s in compact subsets of \mathbb{C} . To do so, break the above into three integrals:

$$\begin{aligned} \int_0^\infty &= \int_0^{\log(m) - \log(m)^{1/2}} \\ &+ \int_{\log(m) - \log(m)^{1/2}}^{\log(m) + \log(m)^{1/2}} + \int_{\log(m) + \log(m)^{1/2}}^\infty. \end{aligned} \quad (10.6)$$

In the middle integral, we have $(x/\log m)^{s-1} - 1 = O(\log(m)^{-1/2})$, with the implied constant uniform for s in compact subsets of \mathbb{C} . Hence, pulling this estimate out, and using, as before,

$$\int_0^\infty (m \exp(-x))^a (1 - \exp(-x))^m dx \sim \Gamma(a) \quad (10.7)$$

we have

$$\begin{aligned} & \int_{\log(m) - \log(m)^{1/2}}^{\log(m) + \log(m)^{1/2}} ((x/\log m)^{s-1} - 1)(m \exp(-x))^a (1 - \exp(-x))^m dx \\ &= O(\log(m)^{-1/2}). \end{aligned} \quad (10.8)$$

Next we bound the first integral, using

$$\begin{aligned} (1 - \exp(-x))^m &= \\ \exp(-m(\exp(-x) + \exp(-2x)/2 + \exp(-3x)/3 + \dots)) \\ &\leq \exp(-m \exp(-x)). \end{aligned} \quad (10.9)$$

Note, further, that $(1 - \exp(-x))^m$ increases monotonically for $x \geq 0$, which, combined with the above estimate gives

$$(1 - \exp(-x))^m \leq \exp(-\exp(\log(m)^{1/2})), \quad (10.10)$$

when $0 \leq x \leq \log(m) - \log(m)^{1/2}$. This shows that $(1 - \exp(-x))^m$ decreases much faster than any power of m , in particular m^a . Furthermore, if $\Re s \geq 1$, then $(x/\log m)^{s-1} - 1$ is bounded over the interval $[0, \log(m) - \log(m)^{1/2}]$. If $\Re s < 1$, then let $\Re s > -M$, where M is a non-negative integer. For $m > M$, we can steal $M + 1$ powers of $(1 - \exp(-x)) = x + O(x^2)$, for $|x| \leq 1$, from $(1 - \exp(-x))^m$, with little effect on that factor, so that $(x/\log m)^{s-1} - 1)(1 - \exp(-x))^{M+1}$ is, on the interval $[0, 1]$ of size $O(\log(m)^{-\sigma+1} + 1)$, where $\sigma = \Re s$. Furthermore, the latter O estimate also holds for $(x/\log m)^{s-1} - 1$ once $x > 1$, if $\Re s < 1$. The integrand of the second integral is thus handily dominated by the last factor of size $O(\exp(-\exp(\log(m)^{1/2})))$ and the second integral tends to 0, uniformly for s in compact subsets of \mathbb{C} .

Finally, to bound the third integral, consider the factor $(x/\log m)^{s-1} - 1$. If $\Re s \leq 1$ then this is bounded for $x \geq \log(m) + \log(m)^{1/2}$. And if $\Re s > 1$ then this factor is $O(x^{\sigma-1}/\log(m)^{\sigma-1})$. Using the trivial bound $(1 - \exp(-x))^m < 1$, we thus have an upper bound for the third integral, when $\Re s \leq 1$, of

$$\ll m^a \int_{\log(m) + \log(m)^{1/2}}^{\infty} \exp(-ax) dx = \frac{\exp(-a \log(m)^{1/2})}{a} \quad (10.11)$$

and, when $\Re s > 1$, of

$$\ll \frac{m^a}{\log(m)^{\sigma-1}} \int_{\log(m) + \log(m)^{1/2}}^{\infty} x^{\sigma-1} \exp(-ax) dx. \quad (10.12)$$

Substituting $u = x - \log(m) - \log(m)^{1/2}$, then pulling out $(\log(m) + \log(m)^{1/2})^{\sigma-1}$ from the integral, and finally using $1 + y < e^y$, with $y = u/(\log(m) + \log(m)^{1/2})$, the above is bounded by

$$\begin{aligned} &\exp(-a \log(m)^{1/2}) \frac{(\log(m) + \log(m)^{1/2})^{\sigma-1}}{\log(m)^{\sigma-1}} \\ &\times \int_0^{\infty} \exp(u(-a + \frac{(\sigma-1)}{(\log(m) + \log(m)^{1/2})})) du \\ &\ll \exp(-a \log(m)^{1/2}), \end{aligned} \quad (10.13)$$

with the implied constant, for given a , uniform for s in compact subsets of \mathbb{C} .

We have thus shown, for given a , that the second integral in (10.4) tends uniformly, for s in compact subsets of \mathbb{C} , to 0, hence proving the asymptotic formula (1.47).

Note that when $s \in \mathbb{Z}$ and $s \leq 0$, then the first line in (10.2), valid for $m > -s$, vanishes because $1/\Gamma(s)$ is then equal to 0. Therefore, the asymptotic formula (1.47) should be interpreted as an equality for such s and m .

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